Logicism - Frege

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Brief Biography

- 1848, born November 8 in Wismar (Mecklenburg-Schwerin)
- 1869, entered the University of Jena
- 1871, entered the University of Göttingen
- 1873, awarded Ph.D. in Mathematics (Geometry), University of Göttingen
- 1879, became Professor Extraordinarius, University of Jena
- 1896, became ordentlicher Honorarprofessor, University of Jena
- 1925, died July 26 in Bad Kleinen (now in Mecklenburg-Vorpommern)
Overview

- Analytic - Synthetic distinction.
- Frege’s theorem: Arithmetic is reduced to logic.
- Ontological commitments.
- Russell’s paradox and impredicative definitions.
Analytic v.s. Synthetic

- Kant’s analytic propositions:
  - A universal proposition of the form All S are P is analytic if P is contained in S, e.g. All bachelors are unmarried.

- Frege’s critique:
  - ‘Kant underestimated the value of analytic judgements ... Taking his definition as a basis, the division of judgements ... is not exhaustive.... How can we do this [determine whether P is contained in S] when the subject is a single object? Or when the judgement is existential?’ (Frege 1884: 107)
Analytic v.s. Synthetic (2)

The distinction Analytic v.s. Synthetic and a priori v.s. a posteriori concern not the content but the justification of a proposition.

If P is justifiable then:
- P is *a priori* iff either it is an unprovable ‘general law’ or a derivation from general laws.
- P is *analytic* iff either it is an unprovable ‘general *logical* law’ or a derivation from general *logical* laws.
Analytic v.s. Synthetic (3)

- A priori
- A posteriori

Analytic

P’s that need in its proof some factual justification

Synthetic A priori
Extension of a Concept

- Extension of a...
  - Singular term
  - General term

The morning star
Phosphorus
Land vehicle s.t. .....
Frege’s Theorem

- **Hume’s principle**: ‘If two numbers are so combined that the one always has a unit which corresponds to each unit of the other, then we claim they are equal’.
- For any concepts F, G the number of F is identical to the number of G if and only if F and G are equinumerous.
- The number of seats in this room is identical with the number of students if each student has a seat and each seat is occupied by a student.
- Hume’s principle doesn’t presuppose the notion of number.
Frege’s Theorem (2)

- $0 :=$ the number of the concept ‘not identical with itself’
- $1 :=$ the number of the concept ‘identical with 0’
- **Successor**: The number $(n+1)$ applies to the concept $F$ if there is an object $a$ which falls under $F$ and such that the number $n$ applies to the concept ‘falling under $F$ but not identical with $a$’.
- $2 :=$ the number of the concept ‘identical with 0 or with 1’
- $n+1 :=$ the number of the concept ‘identical with 0 or 1 or ... or $n$’
- Natural numbers as the smallest inductive set.
The last definition will most quickly arouse hesitation, for, strictly speaking, the sense of the expression ‘the number \( n \) applies to the concept \( G \)’ is just as unknown to us as that of the expression ‘the number \((n+1)\) applies to the concept \( F \)’.

But, to give a crude example, we can never decide by means of our definitions, whether the number \textit{Julius Caesar} applies to a concept ... Furthermore, we cannot prove ... that \( a \) must equal \( b \) if \( a \) applies to the concept \( F \) and \( b \) applies to the same concept.
Frege’s Theorem (4)

- the number which belongs to the concept F is the extension of the concept `equinumerous with the concept F'.
- Hume’s principle follows from the latter definition of number.
- Natural Numbers
  - 0=: \{G: G\ is\ equinumerous\ with\ ‘not\ identical\ with\ itself’\}
  - 1=: \{G: G\ is\ equinumerous\ with\ ‘identical\ with\ 0’\}
  - ...
  - n+1:=\{G: G\ is\ equinumerous\ with\ ‘either\ identical\ with\ 0\ or\ 1\ or...\ or\ n}\
This definition of the number 1 does not presuppose, for its objective legitimacy, any matter of observed fact.

I have already called the attention above to the fact that we say `the number 1' and, by means of the definite article, set up 1 as an object.

Since the important thing here is to grasp the concept of number in such a way that it is useful for science, it needn't disturb us that in everyday usage the number appears attributively.

`Jupiter has four moons' may always be rearranged to form 'The number of Jupiter's moons is four'.

`the number of Jupiter's moons' denotes the same object as the word 'four'.(p.86)
Ontological commitments (2)

- `It was shown that the numbers with which arithmetic concerns itself must be understood not as dependent attributes, but rather substantively. (footnote: The difference corresponds to that between `blue' and `the color of the sky'.) Thus numbers appeared to us as recognizable objects, although not physical ones nor even merely spatial ones, nor ones which we could picture in imagination.' (p. 110)

- Even if the subjective has no spatial location, however, how is it possible for the number 4, which is objective, to be nowhere? Now I maintain that there is no contradiction here. The number 4 is, as a matter of fact, exactly the same for everyone who works with it; but this has nothing to do with spatiality. Not every objective object has a place. (p. 89)
The thesis that principles of arithmetic are derivable from the laws of logic runs against a now common view that logic itself has no ontology.

Frege, however, followed a tradition that concepts are in the purview of logic, and, for Frege, extensions are tied to concepts. So logic does have an ontology. Logical objects include the extensions of some concepts that exist of necessity.

He [Frege] also pointed out that arithmetic enjoys the universal applicability of logic. Any subject-matter has an ontology, and if one has objects at all, one can count them and apply arithmetic.
Assesment of Frege's Theorem

- "It is part of Frege's methodology that one should try to prove what one can, and thus reveal its epistemic ground." (Shapiro 2000: p.112)
- The question is: does Frege's theorem show that arithmetic propositions are either 'general logical laws' or that can be derived from them? (What is a 'general LOGICAL law'?)
- Presuppositions:
  - Hume's principle: (Is it analytic? See neo-logicist discussion)
  - 2nd Order logic: (Does it already embodies too much mathematics on it?)
  - (Basic) set theory:
    - Basic Law V: For any concepts F,G, the extension of F is identical to the extension of G iff for every object a, Fa iff Ga.
Russell’s Paradox

- **Basic Law V:** For any concepts $F, G$, the extension of $F$ is identical to the extension of $G$ iff for every object $a$, $Fa$ iff $Ga$.

- (*) Let $R$ be the concept such that $x$ belongs to the extension of $R$ iff there is a concept $F$ such that $x$ is the extension of $F$ and $x$ is not in the extension of $F$.

- Let $r$ be the extension of $R$.

- If $r$ is in the extension of $R$ then there is a concept $F$ such that $r$ is the extension of $F$ and $r$ is not in the extension of $F$. By Basic Law V, since $F$ and $G$ have the same extension, it follows that $r$ is not in the extension of $R$. (Contradiction)

- If $r$ is not in the extension of $R$ then for all concepts $F$ such that $r$ is the extension of $F$, $r$ is in the extension of $F$. In particular, $r$ is the extension of $R$ and therefore $r$ is in the extension of $R$. (Contradiction)
Russell’s Paradox (2)

- **Basic Law V:** For any concepts F, G, the extension of F is identical to the extension of G iff for every object a, Fa iff Ga.
- (*) Let R be the concept such that x belongs to the extension of R iff there is a concept F such that x is the extension of F and x is not in the extension of F.
- Note that what follows from Basic Law V is just that the concept R so described in (*) cannot exist.
- What is really happening is that Basic Law V is inconsistent with a procedure whatsoever that allows the construction of the set R.
  - In particular allowing all kinds of impredicative definitions,
  - or the Full Comprehension principle, i.e. the possibility of constructing a concept out of an arbitrary proposition.
Russell's Paradox (3)

How to avoid the paradox?

Avoid some construction procedures

- Avoid Impredicative definitions
  - RUSSELL

- Avoid the Full Comprehension Principle
  - ZF

Eliminate Basic Law V

- NEO-LOGICISTS