Cantor (1).

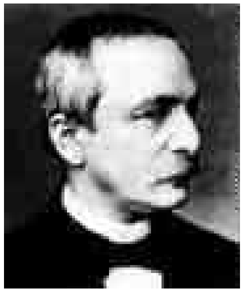
Georg Cantor
(1845-1918)
studied in Zürich, Berlin, Göttingen
Professor in Halle

- Work in analysis leads to the notion of cardinality (1874): most real numbers are transcendental.
- Correspondence with Dedekind (1831-1916): bijection between the line and the plane.
- Perfect sets and iterations of operations lead to a notion of ordinal number (1880).

Cantor (2).

Georg Cantor (1845-1918)

- 1877. Leopold Kronecker (1823-1891) tried to prevent publication of Cantor's work.

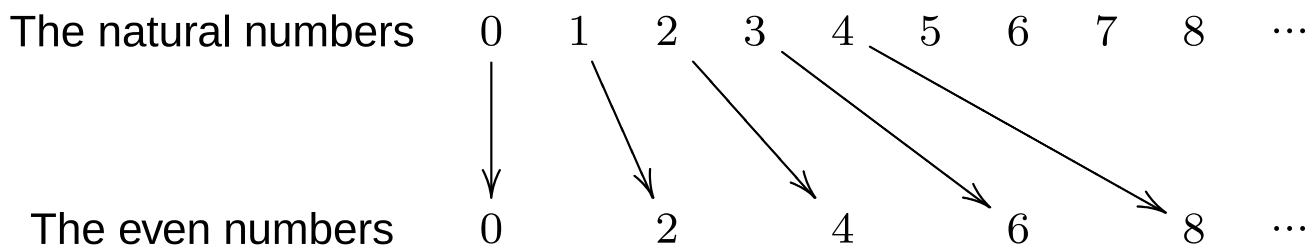


Cantor (2).

Georg Cantor (1845-1918)

- 1877. Leopold Kronecker (1823-1891) tried to prevent publication of Cantor's work.
- Cantor is supported by Dedekind and Felix Klein.
- 1884: Cantor suffers from a severe depression.
- 1888-1891: Cantor is the leading force in the foundation of the *Deutsche Mathematiker-Vereinigung*.
- Development of the foundations of set theory: 1895-1899.

Cardinality (1).



- There is a 1-1 correspondence (bijection) between $\mathbb{N}$ and the even numbers.
- There is a bijection between $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$.
- There is a bijection between $\mathbb{Q}$ and $\mathbb{N}$.
- There is no bijection between the set of infinite 0-1 sequences and $\mathbb{N}$.
- There is no bijection between $\mathbb{R}$ and $\mathbb{N}$.

Cardinality (2).

Theorem (Cantor). There is no bijection between the set of infinite 0-1 sequences and $\mathbb{N}$.

Theorem (Cantor). There is a bijection between the real line and the real plane.

Proof. Let's just do it for the set of infi nite 0-1 sequences and the set of pairs of infi nite 0-1 sequences:
If x is an infi nite 0-1 sequence, then let

$$x_0(n) := x(2n), \text{ and}$$

$$x_1(n) := x(2n+1).$$

Let $F(x) := \langle x_0, x_1 \rangle$. F is a bijection. q.e.d.

Cantor to Dedekind (1877): "*Ich sehe es, aber ich glaube es nicht!*"

Transfiniteness (1).

If $X \subseteq \mathbb{R}$ is a set of reals, we call $x \in X$ isolated in X if no sequence of elements of X converges to x.

Cantor's goal: Given any set X, give a construction of a nonempty subset that doesn't contain any isolated points.

Idea: Let X^{isol} be the set of all points isolated in X, and define $X' := X \backslash X^{\mathrm{isol}}$.

Problem: It could happen that $x \in X'$ was the limit of a sequence of points isolated in X. So it wasn't isolated in X, but is now isolated in X'.

Solution: Iterate the procedure: $X_0 := X$ and $X_{n+1} := (X_n)'$.

Transfiniteness (2).

$X' := X \backslash X^{\mathrm{isol}}$; $X_0 := X$ and $X_{n+1} := (X_n)'$.

Question: Is $\bigcap_{n \in \mathbb{N}} X_n$ a set without isolated points?

Answer: In general, no!

So, you could set $X_\infty := \bigcap_{n \in \mathbb{N}} X_n$, and then $X_{\infty+1} := (X_\infty)'$; in general, $X_{\infty+n+1} := (X_{\infty+n})'$.

The indices used in transfinite iterations like this are called ordinals.

Sets.

The notion of cardinality needs a general notion of function as a special relation between sets. In order to make the notion of an ordinal precise, we also need sets.

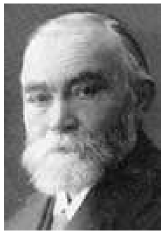
What is a set?

> *Eine Menge ist eine Zusammenfassung bestimmter, wohlunterschiedener Dinge unserer Anschauung oder unseres Denkens zu einem Ganzen.* (Cantor 1895)

The Full Comprehension Scheme. Let X be our universe of discourse ("the universe of sets") and let Φ be any formula. Then the collection of those x such that $\Phi(x)$ holds is a set:

$$\{x\,;\,\Phi(x)\}.$$

Frege (1).



Gottlob Frege (1848-1925)

Frege's Comprehension Principle. If Φ is any formula, then there is some G such that

$$\forall x(G(x) \leftrightarrow \Phi(x)).$$

The ε operator. In Frege's system, we can assign to "concepts" F (second-order objects) a first-order object εF ("the extension of F").

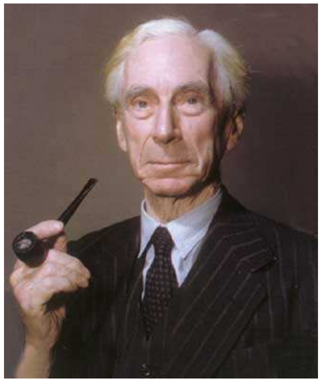
Frege (2).

Basic Law V. If F and G are concepts (second-order objects), then

$$\varepsilon F = \varepsilon G \quad \leftrightarrow \quad \forall x(F(x) \leftrightarrow G(x)).$$

Frege's Foundations of Arithmetic. Let F be an absurd concept ("round square"). Let G be the concept "being equinumerous to εF". We then define $\mathbf{0} := \varepsilon G$.
Suppose $\mathbf{0}$, ... , $\mathbf{n}$ are already defined. Then let H be the concept "being either $\mathbf{0}$ or ... or $\mathbf{n}$" and let $\overline{H}$ be the concept "being equinumerous to εH". Then let $\mathbf{n} + \mathbf{1} := \varepsilon \overline{H}$.

Russell (1).



Bertrand Arthur William
3rd Earl Russell (1872-1970)

- Grandson of John 1st Earl Russell (1792-1878); British prime minister (1846-1852 & 1865-1866).
- 1901: Russell discovers Russell's paradox.
- 1910-13: *Principia Mathematica* with Alfred North Whitehead (1861-1947).
- 1916: Dismissed from Trinity College for anti-war protests.
- 1918: Imprisoned for anti-war protests.
- 1940: Fired from City College New York for anti-war protests.
- 1950: Nobel Prize for Literature.
- 1957: First Pugwash Conference.

Russell (2).

Frege's Comprehension Principle. Every formula defi nes a concept.
Basic Law V. If F and G are concepts, then $\varepsilon F = \varepsilon G \leftrightarrow \forall x(F(x) \leftrightarrow G(x))$.

Theorem (Russell). Basic Law V and the Full Comprehension Principle together are inconsistent.

Proof. Let R be the concept "being the extension of a concept which you don't fall under", *i.e.*, the concept described by the formula

$$\Phi(x) \quad :\equiv \quad \exists F(x = \varepsilon F \ \wedge \ \neg F(x)).$$

This concept exists by **Comprehension**. Let $r := \varepsilon R$.
Either $R(r)$ or $\neg R(r)$:

1. If $R(r)$, then there is some F such that $r = \varepsilon F$ and $\neg F(r)$. Thus $\varepsilon F = \varepsilon R$, and by **Basic Law V**, we have that $F(r) \leftrightarrow R(r)$. But then $\neg R(r)$. Contradiction!
2. If $\neg R(r)$, then for all F such that $r = \varepsilon F$ we have $F(r)$. But R is one of these F, so $R(r)$. Contradiction!

q.e.d.

Russell (3).

Theorem (Russell). The Full Comprehension Principle cannot be an axiom of set theory.

Proof. Suppose the Full Comprehension Principle holds, *i.e.*, every formula Φ describes a set $\{x\,;\,\Phi(x)\}$. Take the formula $\Phi(x) :\equiv x \notin x$ and form the set $r := \{x\,;\,x \notin x\}$ ("the Russell class").
Either $r \in r$ or $r \notin r$.

1. If $r \in r$, then $\Phi(r)$, so $r \notin r$. Contradiction!
2. If $r \notin r$, then $\neg\Phi(r)$, so $\neg r \notin r$, *i.e.*, $r \in r$. Contradiction!

q.e.d.

Frege & Russell.

- Russell discovered the paradox in June 1901.
- Russell's Paradox was discovered independently by Zermelo (Letter to Husserl, dated April 16, 1902).

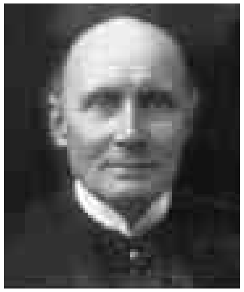
  B. **Rang**, W. **Thomas**, Zermelo's discovery of the "Russell paradox", **Historia Mathematica** 8 (1981), p. 15-22.

- Letter to Frege (June 16, 1902) with the paradox.
- Frege's reply (June 22, 1902): "with the loss of my Rule V, not only the foundations of my arithmetic, but also the sole possible foundations of arithmetic, seem to vanish".

Attempts to resolve the paradoxes.

- **Theory of Types.**
  Russell (1903, “simple theory of types”; 1908, “ramified theory of types”). *Principia Mathematica*.
- **Axiomatization of Set Theory.**
  Zermelo (1908). Skolem/Fraenkel (1922). Von Neumann (1925). “Zermelo-Fraenkel set theory” $\mathbf{ZF}$.
- **Foundations of Mathematics.**
  Hilbert’s 2nd problem: *Consistency proof of arithmetic* (1900). Hilbert’s Programme (1920s).

Principia Mathematica.

 Alfred North Whitehead (1861-1947).

- Mathematician at Trinity College, one of Russell's teachers.
- Continuation of Frege's logicistic programme.
- Later: Philosophy of Science, in particular *Process Ontologies*.

Principia Mathematica: three volumes with a type-theoretic foundations for mathematics; including an axiomatization of arithmetic (1910, 1912, 1913).

Zermelo.

Ernst Zermelo (1871-1953)

- 1894: PhD in Berlin, student of Hermann Amandus Schwarz (1843-1921).
- Assistant of Max Planck, working in hydrodynamics (1894-1897).
- 1904: Proof of the Zermelo Wellordering Theorem (more next week).
- 1905: Professor in Göttingen.
- 1908: Zermelo's Axiom System for Set Theory: Zermelo Set Theory $\mathbf{Z}$.
- 1912: Applications of set theory to mathematical games: Zermelo's Theorem on the determinacy of fi nite games.

Hilbert's Programme (1).

- 1917-1921: Hilbert develops a predecessor of modern first-order logic.
- Paul Bernays (1888-1977)

  - Assistant of Zermelo in Zürich (1912-1916).
  - Assistant of Hilbert in Göttingen (1917-1922).
  - Completeness of propositional logic.
  - "Hilbert-Bernays" (1934-1939).

- Hilbert-Ackermann (1928).
- **Goal.** Axiomatize mathematics and find a finitary consistency proof.

Hilbert's Programme (2).

- 1922: Development of ε-calculus (Hilbert & Bernays). General technique for consistency proofs: "ε-substitution method".
- 1924: Ackermann presents a (false) proof of the consistency of analysis.
- 

  1925: John von Neumann (1903-1957) corrects some errors and proves the consistency of an ε-calculus without the induction scheme.
- 1928: At the ICM in Bologna, Hilbert claims that the work of Ackermann and von Neumann constitutes a proof of the consistency of arithmetic.

Brouwer (1).

L. E. J. (Luitzen Egbertus Jan) Brouwer
(1881-1966)

- Student of Korteweg at the UvA.
- 1909-1913: Development of topology. Brouwer's Fixed Point Theorem.
- 1913: Succeeds Korteweg as full professor at the UvA.
- 1918: *"Begründung der Mengenlehre unabhängig vom Satz des ausgeschlossenen Dritten"*.

Brouwer (2).

- 1920: *“Besitzt jede reelle Zahl eine Dezimalbruch-Entwickelung?”*. Start of the *Grundlagenstreit*.
- 
  1921: Hermann Weyl (1885-1955), *“Über die neue Grundlagenkrise der Mathematik”*
- 1922: Hilbert, *“Neubegründung der Mathematik”*.
- 1928-1929: ICM in Bologna; *Annalenstreit*. Einstein and Carathéodory support Brouwer against Hilbert.

Intuitionism.

- Constructive interpretation of existential quantifiers.
- As a consequence, rejection of the *tertium non datur*.
- More in the guest lecture on November 17.
- The big three schools of philosophy of mathematics: logicism, formalism, and intuitionism.
- Nowadays, different positions in the philosophy of mathematics are distinguished according to their view on ontology and epistemology. Main positions are: (various brands of) Platonism, Social Constructivism, Structuralism, Formalism.

Gödel (1).

Kurt Gödel (1906-1978)

- Studied at the University of Vienna; PhD supervisor Hans Hahn (1879-1934).
- Thesis (1929): Gödel Completeness Theorem.
- 1931: "*Über formal unentscheidbare Sätze der* Principia Mathematica *und verwandter Systeme I*". Gödel's First Incompleteness Theorem and a proof sketch of the Second Incompleteness Theorem.

Gödel (2).

- 1935-1940: Gödel proves the consistency of the Axiom of Choice and the Generalized Continuum Hypothesis with the axioms of set theory (solving one half of Hilbert's 1st Problem).
- 1940: Emigration to the USA: Princeton.
- Close friendship to Einstein, Morgenstern and von Neumann.
- Suffered from severe hypochondria and paranoia.
- Strong views on the philosophy of mathematics.

Gödel's Incompleteness Theorem (1).

1928: At the ICM in Bologna, Hilbert claims that the work of Ackermann and von Neumann constitutes a proof of the consistency of arithmetic.

- 1930: Gödel announces his result (G1) in Königsberg in von Neumann's presence.
- Von Neumann independently derives the Second Incompleteness Theorem (G2) as a corollary.
- Letter by Bernays to Gödel (January 1931): There may be finitary methods not formalizable in $\mathbf{PA}$.
- 1931: Hilbert suggests new rules to avoid Gödel's result. Finitary versions of the ω-rule.
- By 1934, Hilbert's programme in the original formulation has been declared dead.

Gödel's Incompleteness Theorem (2).

Theorem (Gödel's Second Incompleteness Theorem). If T is a consistent axiomatizable theory containing $\mathbf{PA}$, then $T \nvdash \mathrm{Cons}(T)$.

- "consistent": $T \nvdash \bot$.
- "axiomatizable": T can be listed by a computer ("computably enumerable", "recursively enumerable").
- "containing $\mathbf{PA}$": $T \vdash \mathbf{PA}$.
- "$\mathrm{Cons}(T)$": The formalized version (in the language of arithmetic) of the statement 'for all T-proofs P, $\bot$ doesn't occur in P'.

Gödel's Incompleteness Theorem (3).

- Thus: Either PA is inconsistent or the deductive closure of PA is not a complete theory.
- All three conditions are necessary:
  - **Theorem** (Presburger, 1929). There is a weak system of arithmetic that proves its own consistency ("Presburger arithmetic").

Mojzesz Presburger (1904-c. 1943)

Gödel's Incompleteness Theorem (3).

- Thus: Either $\mathbf{PA}$ is inconsistent or the deductive closure of $\mathbf{PA}$ is not a complete theory.
- All three conditions are necessary:
  - **Theorem** (Presburger, 1929). There is a weak system of arithmetic that proves its own consistency ("Presburger arithmetic").
  - If T is inconsistent, then $T \vdash \varphi$ for all φ.
  - If $\mathbb{N}$ is the standard model of the natural numbers, then $\mathrm{Th}(\mathbb{N})$ is a complete extension of $\mathbf{PA}$ (but not axiomatizable).

Gentzen.

Gerhard Gentzen (1909-1945)

- Student of Hermann Weyl (1933).
- 1934: Hilbert's assistant in Göttingen.
- 1934: Introduction of the Sequent Calculus.
- 1936: Proof of the consistency of $\mathbf{PA}$ from a transfinite wellfoundedness principle.

  Theorem (Gentzen). Let $T \supseteq \mathbf{PA}$ such that T proves the existence and wellfoundedness of (a code for) the ordinal ε_0. Then $T \vdash \mathrm{Cons}(\mathbf{PA})$.