UNIVERSITEIT VAN AMSTERDAM
INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

# Axiomatic Set Theory

**(Axiomatische Verzamelingentheorie)**

**2002/2003; 3rd Trimester**
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**Homework Set # 7 [Jech, § 10 and § 11]** Deadline: Thursday, July 3rd, 2003

**Exercise 7.1** (Normal Measures)

Recall that we call an ultrafilter $U$ on $\kappa$ a **normal measure** if for every $X \in U$ and every regressive function $f : X \to \kappa$ there is a $Y \in U$ such that $f$ is constant on $Y$.
Show that an ultrafilter is a normal measure if and only if it is closed under diagonal intersections.

**Exercise 7.2** ($G_\delta$ and $F_\sigma$ sets)

The $G_\delta$ sets are countable intersections of open sets of reals, the $F_\sigma$ sets are countable unions of closed sets of reals.
Show that if $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, then $f$-preimages of $G_\delta$ ($F_\sigma$) sets are $G_\delta$ ($F_\sigma$).
What properties of the real numbers did you use?

**Exercise 7.3** (How many Borel sets are there?)
Prove that there are $2^{\aleph_0}$ closed sets.

**Hint.** To get an upper bound, use the tree representation of closed sets. How many trees are there? To get a lower bound, think of very simple closed sets.

Prove (using the Axiom of Choice) that there are $2^{\aleph_0}$ Borel sets.

**Hint.** Prove it by induction on the level of the Borel hierarchy.

**Exercise 7.4** (The Separation Property)
Let $\Gamma$ be a class of sets. We say that $\Gamma$ **has the separation property** if the following holds:

> For every two disjoint sets $A$ and $B$ in $\Gamma$ there is a set $C$ such that both $C$ and the complement of $C$ are in $\Gamma$ with $A \subseteq C$ and $B \cap C = \varnothing$.

Show that the class $\mathbf{\Sigma}^1_1$ of analytic sets has the separation property.