UNIVERSITEIT VAN AMSTERDAM
INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

Axiomatic Set Theory

(Axiomatische Verzamelingentheorie)

2002/2003; 3rd Trimester
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Homework Set # 5 [Jech, § 5 and § 6] Deadline: Thursday, June 12th, 2003

For exercises # 5.1 and # 5.2, use the "inductive computation of κ^λ" (Theorem 5.20 in Jech's book):
Cardinal exponentiation κ^λ can be computed by induction on κ as follows:

(1) If $\kappa \leq \lambda$, then $\kappa^\lambda = 2^\lambda$.
(2) If there is some $\mu < \kappa$ such that $\mu^\lambda \geq \kappa$, then $\kappa^\lambda = \mu^\lambda$.
(3) If $\kappa > \lambda$ and $\mu^\lambda < \kappa$ for all $\mu < \kappa$, then
  (a) if $\mathrm{cf}(\kappa) > \lambda$, then $\kappa^\lambda = \kappa$, and
  (b) if $\mathrm{cf}(\kappa) \leq \lambda$, then $\kappa^\lambda = \kappa^{\mathrm{cf}(\kappa)}$.

Exercise 5.1 (Cardinal Arithmetic with exponent $\aleph_1$)

(1) Show that $(\aleph_n)^{\aleph_1} = \aleph_n \cdot 2^{\aleph_1}$.
(2) Show that $(\aleph_\omega)^{\aleph_1} = (\aleph_\omega)^{\aleph_0} \cdot 2^{\aleph_1}$.
(3) Show that $(\aleph_{\omega+1})^{\aleph_1} = (\aleph_{\omega+1})^{\aleph_0} \cdot 2^{\aleph_1}$.

Exercise 5.2 (Cardinal Arithmetic under the assumption $2^{\aleph_1} = \aleph_2$)
In this exercise, we work in $\mathsf{ZFC} + 2^{\aleph_1} = \aleph_2$. Keep the computations of # 5.1 in mind.

(1) Compute $(\aleph_n)^{\aleph_1}$ for all values of $n \in \omega$. ("Computation" means: determine α such that $(\aleph_n)^{\aleph_1} = \aleph_\alpha$.)
(2) Show that $(\aleph_\omega)^{\aleph_1} = (\aleph_\omega)^{\aleph_0}$.
(3) Show that $(\aleph_\omega)^{\aleph_0} \neq \aleph_{\aleph_1}$. (**Hint.** What is $((\aleph_\omega)^{\aleph_0})^{\aleph_1}$? What is $(\aleph_{\aleph_1})^{\aleph_1}$?)
(4) Assume in addition that $(\aleph_\omega)^{\aleph_0} > \aleph_{\aleph_1}$. Show that under this additional assumption, we get $(\aleph_\omega)^{\aleph_0} = (\aleph_{\aleph_1})^{\aleph_1}$.

Exercise 5.3 (Mirimanoff rank)
Let ϱ the Mirimanoff rank. Show that for every set x, we have

$$\varrho(x) = \sup\{\varrho(z) + 1\,;\, z \in x\}.$$

Exercise 5.4 (Singular ordinals and the Axiom of Replacement)
We call α **definably singularized** if it has the following property: There is an ordinal $\beta < \alpha$ and formula Ψ such that

(1) $\Psi(x)$ implies that $x = \langle \gamma, \delta \rangle$ for some $\gamma < \beta$ and $\delta < \alpha$.
(2) Ψ is functional, *i.e.*, $\Psi(\langle \gamma, \delta \rangle)$ and $\Psi(\langle \gamma, \delta^* \rangle)$ imply that $\delta = \delta^*$.
(3) For each $\eta < \alpha$ there is some $\gamma < \beta$ and δ with $\Phi(\gamma, \delta)$ such that $\delta > \eta$.

Prove that if α is a definably singularized limit ordinal, then $\mathbf{V}_\alpha$ is not a model of the Replacement Axiom (Ers).
Hint. Suppose that $\mathbf{V}_\alpha$ satisfies (Ers) for a contradiction, and use Ψ to prove that $\alpha \in \mathbf{V}_\alpha$.