UNIVERSITEIT VAN AMSTERDAM
INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

Axiomatic Set Theory

(Axiomatische Verzamelingentheorie)

2002/2003; 3rd Trimester
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Homework Set # 4 Deadline: Monday, June 2nd, 2003

Exercise 4.1 (Mathematical Constructions in Set Theory)

In the lecture course, we defined the set $\mathbb{Z}$ as $\{0\} \times (\mathbb{N}\backslash\{0\}) \cup \{1\} \times \mathbb{N}$, addition and multiplication on $\mathbb{Z}$, the set $\mathbb{Q}$ as the set of $\sim_{\mathbb{Q}}$ equivalence classes where $\sim_{\mathbb{Q}}$ is the equivalence relation on $\mathbb{Z} \times \mathbb{N}\backslash\{0\}$ defined by

$$\langle z, n\rangle \sim_{\mathbb{Q}} \langle w, m\rangle \iff z \cdot m = w \cdot n.$$

We further defined addition and multiplication on $\mathbb{Q}$ and the set of positive rational numbers, and used that to define the ordering on $\mathbb{Q}$. We then defined the set of Dedekind cuts of $\mathbb{Q}$ and defined an ordering on them.

(1) Go through these steps of construction and discuss what axioms of set theory are used in each of them. How much set theory do we need to prove that the ordered structure of the real numbers is a set?

(2) If $\langle P, <\rangle$ is any linearly ordered set, prove that $\langle D(P), <_{\mathrm{D}}\rangle$ is a completion of P in the following sense:
(a) For all $d_0 \neq d_1 \in D(P)$ there is a $p \in P$ such that $d_0 <_{\mathrm{D}} \overline{p} <_{\mathrm{D}} d_1$ where

$$\overline{p} := \langle\{q\,;\, q \leq p\}, \{q\,;\, q > p\}\rangle, \text{ and}$$

(b) for every bounded set $B \subseteq P$, the set $\overline{B} := \{\overline{p}\,;\, p \in B\}$ has a least upper bound in $D(P)$.

Exercise 4.2 (The Wellordering Theorem)

Work in ZF (without the Axiom of Choice). Prove that the statement of the wellordering theorem ("every set is in bijection with an ordinal") implies the Axiom of Choice.

Hint. Trying to give a choice function for the set (of sets) X, wellorder $\bigcup X$ by $f : \bigcup X \to \alpha$ for some ordinal α, and then define a function $F : X \to \bigcup X$ by

$$\langle x, y\rangle \in F :\iff y \in x \;\&\; \forall z(z \in x \to f(y) \leq f(z)).$$

Exercise 4.3 (Abstract Bernstein Theorem)

Prove (using the Axiom of Choice) the following **Abstract Bernstein Theorem**: If X is a set of cardinality κ and $\langle Y_\alpha\,;\, \alpha < \kappa\rangle$ is a sequence of sets of cardinality κ, then there is a set $A \subseteq X$ such that both A and $X\backslash A$ intersect each of the Y_α.

Exercise 4.4 (Cantor-Bendixson number)

Give examples of subsets of $\mathbb{R}$ that have Cantor-Bendixson number 3 and 4.

Exercise 4.5 (Countable Ordinals and the Reals)

Show that every countable ordinal can be injected order-preservingly into the real numbers.

Hint. Induction on α. Note that there is an order-preserving bijection between $\mathbb{R}$ and the open interval $(n, n+1)$, so every ordinal that can be injected into $\mathbb{R}$ can be injected into a bounded subset of $\mathbb{R}$.