UNIVERSITEIT VAN AMSTERDAM
INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

Axiomatic Set Theory
(Axiomatische Verzamelingentheorie)

2002/2003; 3rd Trimester
dr Benedikt Löwe

Homework Set # 3 Deadline: Monday, May 17th, 2003

Exercise 3.1 (Countable Sets)
A set A is called **countable** if there is an injection from A to the set of natural numbers $\mathbb{N}$.

(1) Show: If C is countable and $A \subseteq C$, then A is countable.
(2) Show: If $A_0, \ldots, A_n$ are finitely many countable sets, then $A_0 \cup \ldots \cup A_n$ is countable.
(3) Let $F = \langle f_i\,;\, i \in \mathbb{N}\rangle$ be a sequence of injective functions where $f_i : A_i \to \mathbb{N}$ (in particular, this means that A_i is countable). Show that $\bigcup\{A_i\,;\, i \in \mathbb{N}\}$ is countable.
(4) Why did we formulate (3) in that complicated way? Why is the existence of F (as opposed to the mere existence of the f_i) so important? (**Hint.** It has to do with the Axiom of Choice.)
(5) Show that the set of all finite sequences of natural numbers is countable.

Exercise 3.2 ($\aleph_1$ and $2^{\aleph_0}$)
For $A \subseteq \omega$ define
$$R_A := \{\langle n, m\rangle\,;\, \ulcorner n, m\urcorner \in A\}$$
where $\ulcorner .,. \urcorner : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is the bijection from Exercise 0.3.
If $\langle \mathbb{N}, R_A\rangle$ is a wellordered set, let α_A be its ordertype.
Show that $A \mapsto \begin{cases} \alpha_A & \text{if defined,} \\ 0 & \text{otherwise} \end{cases}$ is a surjection from $2^{\aleph_0}$ onto $\aleph_1$.

Exercise 3.3 (ω_2 and countable unions)
Show that ω_2 cannot be the countable union of countable sets.
(**Hint.** Each countable subset $S \subseteq \omega_2$ has an ordertype $< \omega_1$. Use this to construct a surjection from $\omega \times \omega_1$ onto ω_2 and derive a contradiction.)

Exercise 3.4 ($\aleph$- and $\beth$-fixed points)
A cardinal κ is called an **$\aleph$-fixed point** (**$\beth$-fixed point**) if $\aleph_\kappa = \kappa$ ($\beth_\kappa = \kappa$).
Show that each $\beth$-fixed point is an $\aleph$-fixed point.
Show that there are arbitrarily large $\beth$-fixed points (*i.e.*, for each $\lambda \in \mathrm{Ord}$ there is a $\beth$-fixed point larger than λ).
Describe the least $\beth$-fixed point and compute its cofinality.
(**Hint.** Use Exercise 2.2.)