UNIVERSITEIT VAN AMSTERDAM
INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

Axiomatic Set Theory

(Axiomatische Verzamelingentheorie)

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Homework Set # 0 Deadline: April 17th, 2003

Exercise 0.1

Let R be any binary relation on $X \times Y$, *i.e.*, $R \subseteq X \times Y$. Let $A \subseteq X$, then call

$$R[A] := \{y\,;\, \exists a \in A(\langle a, y\rangle \in R)\}$$

the **image of** A **under** R. Similarly, for $B \subseteq Y$, call

$$R^{-1}[B] := \{x\,;\, \exists b \in B(\langle x, b\rangle \in R)\}$$

the **preimage of** B **under** R.

Let now $\{A_n\,;\, n \in \mathbb{N}\}$ be a family of subsets of X and $\{B_n\,;\, n \in \mathbb{N}\}$ be a family of subsets of Y. Consider the following set equalities:

$$R^{-1}[\bigcup_{n=0}^{\infty} B_n] = \bigcup_{n=0}^{\infty} R^{-1}[B_n], \quad R^{-1}[\bigcap_{n=0}^{\infty} B_n] = \bigcap_{n=0}^{\infty} R^{-1}[B_n],$$

$$R[\bigcup_{n=0}^{\infty} A_n] = \bigcup_{n=0}^{\infty} R[A_n], \quad R[\bigcap_{n=0}^{\infty} A_n] = \bigcap_{n=0}^{\infty} R[A_n].$$

Which of them hold for all relations, which hold only for functions, and which hold only for injections? Does this have anything to do with quantifier logic?

Exercise 0.2

Show that the relation $\simeq$ of **equinumerosity** ($A \simeq B$ if and only if there is a bijection from A to B) is an equivalence relation (*i.e.*, reflexive, symmetric and transitive).

Exercise 0.3
Show that the function $\ulcorner \cdot, \cdot \urcorner : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by

$$\ulcorner i, j \urcorner := \frac{i^2 + j^2 + 2ij + i + 3j}{2}$$

is a bijection from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$.
Define a function on the set $\mathbb{Q}$ of fractions as follows. For $q \in \mathbb{Q}$ take n_q and d_q such that $q = \frac{n_q}{d_q}$ and d_q does not divide n_q. (The fraction is in fully reduced form.) Then let $i(q)$ be the least natural number k such that $k = \ulcorner n_q, d_q \urcorner$. Prove that the function i is an injection from $\mathbb{Q}$ to $\mathbb{N}$.

Exercise 0.4
Show that the set operation

$$\langle A_i \, ; \, i \in \mathbb{N} \rangle \mapsto \bigcup_{n=0}^{\infty} A_n$$

is a special case of the general union operation

$$X \mapsto \bigcup X := \{\xi \, ; \, \exists x \in X (\xi \in x)\}$$

by defining some set X^* (depending on $\langle A_i \, ; \, i \in \mathbb{N} \rangle$) such that

$$\bigcup X = \bigcup_{n=0}^{\infty} A_n.$$