June 9, 2020

Differential Geometry

lecture notes, summer term 2020, University of Hamburg

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Contents

1 Smooth manifolds and vector bundles

1.1 Basic definitions

In the field of differential geometry one is concerned with geometric objects that look locally like $\mathbb{R}^n$ for some $n \in \mathbb{N}$. In the following we will clarify exactly what this should mean and explain the reason for the term "differential" in differential geometry.

Remark 1.1. Recall the definition of a **topological space**. Let M, N be topological spaces. A map $f : M \to N$ is called **continuous** if for all $U \subset N$ open, $f^{-1}(U) \subset M$ is open. A continuous map is called a **homeomorphism** if it has an inverse, that is if it is bijective as a map between sets, and the inverse is continuous. A **basis of the topology** of a topological space M is a collection of open sets $\mathcal{B}$, so that for all $U \subset M$ open there exist an index set I and corresponding open sets B_i each contained in $\mathcal{B}$, such that $\cup_{i \in I} B_i = B$. Note that I might be uncountable.

The study of topological spaces in full generality is not the topic of this course. We need to introduce two additional properties that topological spaces might fulfil in order to define the kind objects will study, namely smooth manifolds.

Definition 1.2. Let M be a topological space. M is called **Hausdorff**[1] if for any two distinct points $p, q \in M$, $p \neq q$, we can find $U, V \subset M$ open, such that $p \in U$, $q \in V$, and $U \cap V$. This

[1]Felix Hausdorff (1868 – 1942)

means that we can separate any distinct points in M with disjoint open sets. M is said to fulfil the **second countability axiom** (or, simply, are second countable) if its topology has a countable basis.

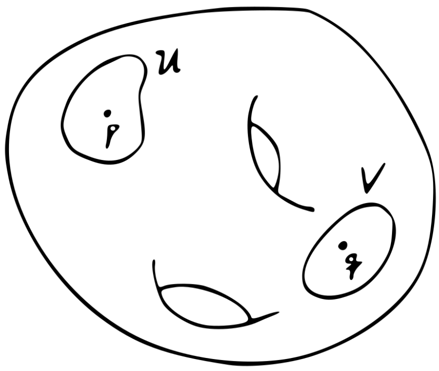


Figure 1: Open sets U and V in a Hausdorff space separating two points p and q.

If the reader is new to general topology and the above definitions seem confusing, consider the following well known examples of Hausdorff topological spaces that are second countable. These are also basically the only examples the reader has to keep in mind for this course.

Example 1.3. For any $n \in \mathbb{N}_0$, $\mathbb{R}^n$ equipped with its standard topology induced by the Euclidean norm is Hausdorff and second countable. A choice for a countable basis of the topology is given by

$$\mathcal{B} := \{B_r(p) \mid r \in \mathbb{Q}_{>0},\ p \in \mathbb{Q}^n\}\,.$$

[Exercise: Prove that $\mathcal{B}$ is in fact a basis of the norm topology on $\mathbb{R}^n$.]

Now we have introduced all topological perquisites. Next, we will give a precise meaning to the term "locally looks like" that we have used before

Definition 1.4. Let M be a Hausdorff topological space that is second countable. An **n-dimensional smooth atlas on** M,

$$\mathcal{A} = \{(\varphi_i, U_i) \mid i \in A\}\,,$$

is a collection of tuples (φ_i, U_i), each consisting of an open set $U_i \subset M$ and a homeomorphism

$$\varphi_i : U_i \to \varphi_i(U_i) \subset \mathbb{R}^n, \tag{1.1}$$

such that

(i) $\bigcup_{i \in A} U_i = M$, that is the U_i form a covering of M,

(ii)
$$\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j) \tag{1.2}$$
is smooth for all $i, j \in A$ with $U_i \cap U_j \neq \emptyset$.

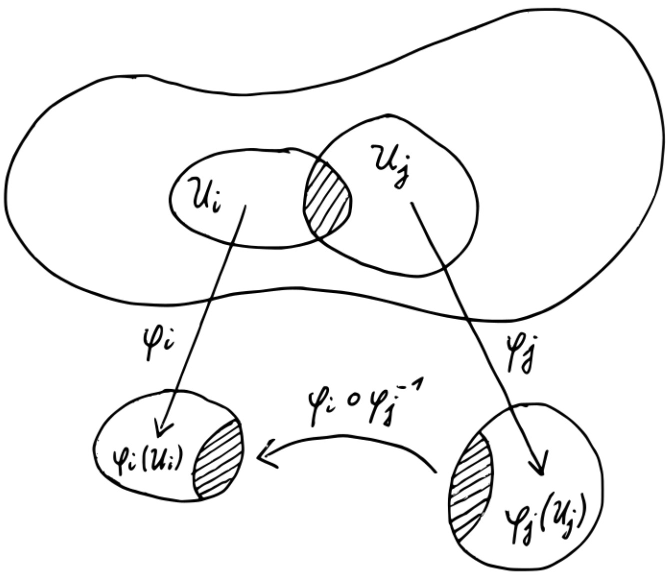


Figure 2: Two charts (φ_i, U_i) and (φ_j, U_j) with $U_i \cap U_j \neq \emptyset$.

Maps of the form (1.1) together with their domains are called **charts** on M and the maps in (1.2) are corresponding **transition functions**. Any two charts (φ_i, U_i), (φ_j, U_j), not necessarily from the same atlas, are called **compatible** if the corresponding transition function $\varphi_i \circ \varphi_j^{-1}$ and its inverse are smooth.

In the following we will simply speak of atlases and drop the prefix "n-dimensional smooth", unless it is of specific value for a statement. Now consider the following questions. Firstly assume that you are given two different atlases $\mathcal{A}$ and $\mathcal{B}$ on M. What is a good notion for compatibility of these two atlases? A reasonable idea is to require that their charts are compatible in the sense of (1.2). Secondly there should always be the question whether or not there is a canonical choice for some sort of structure, in this setting that of an atlas. This leads us to the following definition:

Definition 1.5. Two atlases $\mathcal{A} = \{(\varphi_i, U_i) \mid i \in A\}$ and $\mathcal{B}$ $\{(\varphi_i, U_i) \mid i \in B\}$ on a second countable Hausdorff topological space M are called **equivalent** if

$$\mathcal{A} \cup \mathcal{B} := \{(\varphi_i, U_i) \mid i \in A \cup B\}$$

is an atlas on M. This is equivalent to the requirement that the transition function $\varphi_i \circ \varphi_j^{-1}$ (1.2) for all $i, j \in A \cup B$ are smooth. For $\mathcal{A}$ and $\mathcal{B}$ equivalent we write $[\mathcal{A}] = [\mathcal{B}]$. An atlas $\overline{\mathcal{A}}$ on M is called **maximal** if for all atlases $\mathcal{A}'$ on M equivalent to $\overline{\mathcal{A}}$ it holds that $\mathcal{A}' \subset \mathcal{A}$.

Now we have all tools at hand to define the notion of a smooth manifold:

Definition 1.6. A second countable Hausdorff topological space M together with a maximal n-dimensional smooth atlas $\mathcal{A}$ is called an **smooth manifold** of dimension n.

In the following we will always assume that smooth manifolds are of dimension $n \geq 1$.

Remark 1.7. If one left out the requirement of second countability, the definition of a smooth manifold would still be usable for effectively every local statement about smooth manifolds. This approach is for example taken in [O]. However some global constructions might not work, in particular those involving a countable partition of unity (cf. Exercise **??**) which might not exist. An example of an analogue of a smooth manifold that is not second countable is the so-called "long line" [SS].

We will call the process of defining a maximal atlas on M, defining the structure of a smooth manifold on M. A caveat of the above definition is that it is not in any way clear how to completely specify or write down a maximal atlas, at least not if $n > 0$. The following lemma deals with this problem.

Lemma 1.8. Let $\mathcal{A}$ be an atlas on a second countable Hausdorff topological space M. Then $\mathcal{A}$ is contained in a maximal atlas, i.e. there exists a maximal atlas $\overline{\mathcal{A}}$ on M, such that $\mathcal{A} \subset \overline{\mathcal{A}}$.

Proof. The set of atlases equivalent to $\mathcal{A}$, $\mathrm{Eq}(\mathcal{A})$, is a partially ordered ordered set with respect to the inclusion. By Zorn's[2] lemma $\mathrm{Eq}(\mathcal{A})$ contains a maximal element $\overline{\mathcal{A}}$ which by construction is an atlas and fulfils all requirements of a maximal atlas. □

Remark 1.9. The precise statement of Zorn's lemma is that every partially ordered set $(S, <)$ has a maximal element. This means that there exists $s_{\max} \in S$, such that either $s < s_{\max}$, or neither $s < s_{\max}$ nor $s > s_{\max}$. Note that $s_{\max}$ is in general **not unique**.

Remark 1.9 raises the question whether a maximal atlas containing any given atlas is uniquely determined. The answer is yes, and the proof feels a bit like we were cheating.

Lemma 1.10. Each atlas is contained in a unique maximal atlas.

Proof. Let $\mathcal{A} = \{(\varphi_i, U_i) \mid i \in A\}$ be an n-dimensional smooth atlas on a second countable Hausdorff topological space M. We define

$$\overline{\mathcal{A}} := \{(\varphi, U) \mid \varphi : U \to \varphi(U) \text{ is a chart on } M,\, \varphi \text{ and } \varphi_i \text{ are compatible } \forall i \in A\}.$$

We now write $\overline{\mathcal{A}} = \left\{(\varphi_i, U_i) \;\middle|\; i \in \overline{A}\right\}$ and claim that it is both a maximal atlas and unique in the stated sense. Firstly note that $\mathcal{A} \subset \overline{\mathcal{A}}$ and, hence, $\bigcup_{i \in \overline{A}} U_i = M$. Next we need to show that for any $i, j \in \overline{A}$, $\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$ is smooth. Being smooth is a local property, so we fix any point $p \in \varphi_j(U_i \cap U_j)$ and choose a chart (φ, U) in $\mathcal{A}$, such that $p \in \varphi(U)$. Then we choose $V \subset \varphi(U) \cap \varphi_j(U_i \cap U_j)$, $V \subset \mathbb{R}^n$ open, such that $p \in \varphi^{-1}(V)$, observe that

$$\varphi_i \circ \varphi_j^{-1} = (\varphi_i \circ \varphi^{-1}) \circ (\varphi \circ \varphi_j^{-1})$$

coincide on V. Since the right-hand-side of the above equation is a composition of by construction of $\overline{\mathcal{A}}$ smooth maps, it follows that $\varphi_i \circ \varphi_j^{-1}$ is smooth as well. This shows that $\overline{\mathcal{A}}$ is indeed an n-dimensional smooth atlas on M and that $\mathcal{A} \subset \overline{\mathcal{A}}$. Lastly assume that $\overline{\mathcal{A}}$ is not maximal. Then there exists an atlas $\mathcal{A}'$ on M that is equivalent to $\overline{\mathcal{A}}$ and there exists a chart (φ, U) in $\mathcal{A}'$ that is not contained in $\overline{\mathcal{A}}$. By $\mathcal{A} \subset \overline{\mathcal{A}}$ this means that even though (φ, U) is compatible with every chart in $\mathcal{A}$ it is not contained in $\overline{\mathcal{A}}$. This is a contradiction to the construction of $\overline{\mathcal{A}}$. This shows that $\overline{\mathcal{A}}$ is maximal and finishes the proof. □

Remark 1.11. We have seen in Lemma 1.8 that it is **sufficient** to specify an atlas $\mathcal{A}$ on a second countable Hausdorff topological space M in order to define the structure of a smooth manifold on M without the need of requiring maximality of $\mathcal{A}$. Furthermore, we have proven in Lemma 1.10 that there exists a **unique** maximal atlas on M that is equivalent to $\mathcal{A}$, meaning that there is no possibility to choose an other structure of a smooth manifold on M for which $\mathcal{A}$ is an atlas. This justifies calling a second countable Hausdorff topological space equipped with **any** atlas, be it maximal or not, a smooth manifold.

[2] Max August Zorn (1906 – 1993)

It is however not clear at this point whether for a given smooth manifold M with maximal atlas $\overline{\mathcal{A}}$ there might exist some other maximal atlas $\overline{\mathcal{B}}$ on the underlying topological space M that is not equivalent to $\overline{\mathcal{A}}$. This is in general a very difficult question. There are some examples where this question has been answered, see the so-called exotic spheres [M] and for 4-dimensional smooth manifolds [Sc].

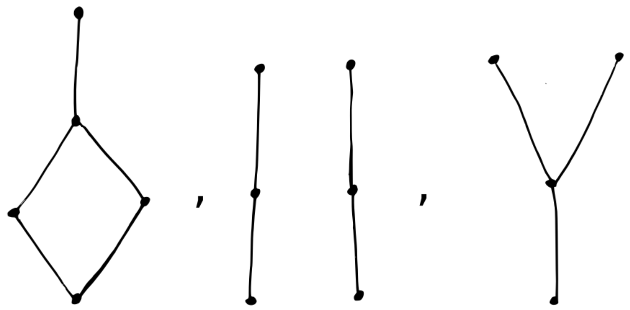


Figure 3: Which of the three partially ordered sets (higher means $\geq$) is a good representation for the equivalence classes of atlases?

Next we should ask ourselves how "good" we might expect a choice of an atlas for a given smooth manifold to look like, meaning an atlas contained in the by definition provided maximal atlas. Can we always choose a countable atlas, that is an atlas containing only a countable number of charts, that is equivalent to our given maximal atlas? Can we always choose a finite atlas if our manifold is connected? The answer is yes to both, but the latter is much more difficult to prove than the former, for the non-compact case see [So].

Exercise 1.12.

(i) Show that every smooth manifold M with maximal atlas $\overline{\mathcal{A}}$ has a countable atlas that is equivalent to $\overline{\mathcal{A}}$. [Hint: Use that M is second countable.]

(ii) Show that every connected compact smooth manifold with maximal atlas $\overline{\mathcal{A}}$ has a finite atlas that is equivalent to $\mathcal{A}$.

An important analytical tool that we will need in this course is to shrink the chart neighbourhoods of a given atlas.

Definition 1.13. Let $\mathcal{A} = \{(\varphi_i, U_i) \mid i \in A\}$ be an atlas on a smooth manifold M. Another atlas on M, $\widetilde{\mathcal{A}} = \{(\widetilde{\varphi}_i, \widetilde{U}_i) \mid i \in \widetilde{A}\}$, will be called a **refinement of** $\mathcal{A}$ if for all $i \in \widetilde{A}$ there exists $j \in A$, such that $\widetilde{U}_i \subset U_j$ and $\widetilde{\varphi}_i = \varphi_j|_{\widetilde{U}_i}$.

Exercise 1.14. Check that any atlas is equivalent to any possible refinement of itself.

We have now finished setting up the theoretical framework for the basic definitions of smooth manifolds, so next we should study some examples.

Example 1.15. The probably easiest example of a smooth manifold is $\mathbb{R}^n$ equipped with the atlas containing the sole chart $(\mathrm{id}, \mathbb{R}^n)$ containing only the identity map

$$\mathrm{id} = (u^1, \ldots, u^n), \quad u^i : (p_1, \ldots, p_n) \mapsto p_i, \tag{1.3}$$

with domain the whole $\mathbb{R}^n$. It is also immediate that for any $U \subset \mathbb{R}^n$ open, U equipped with (id, U) is also a smooth manifold.

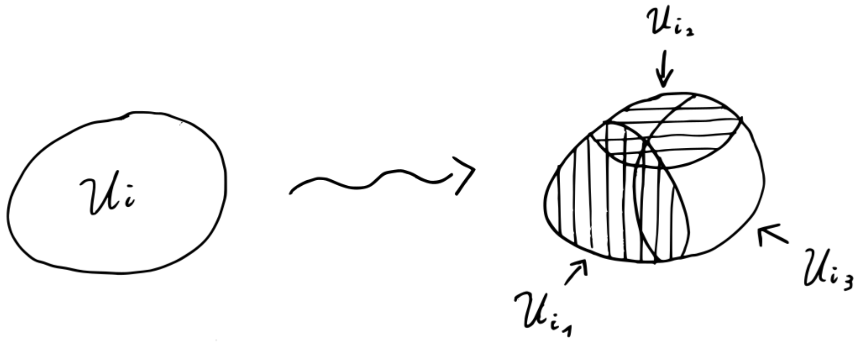


Figure 4: Refining a chart neighbourhood U_i into three proper subsets U_{i_1}, U_{i_2}, and U_{i_3}.

Definition 1.16. The maps u^i, $1 \leq i \leq n$, in (1.3) are called **canonical coordinates** on any open subset $U \subset \mathbb{R}^n$.

A similar notation is used for general smooth manifolds.

Definition 1.17. Let M be an n-dimensional smooth manifold and (φ, U) be a chart on M. With the notation

$$\varphi = (u^1 \circ \varphi, \dots, u^n \circ \varphi), \quad u^i \circ \varphi : U \to \mathbb{R},\ 1 \leq i \leq n,$$

the maps $x^i := u^i \circ \varphi$, $1 \leq i \leq n$, are called **local coordinates** on M, and φ is called **local coordinate system**.

Now that we have setup our basic theoretical framework, it is time to look at some non-trivial examples of smooth manifolds to get a better feeling for what one needs to validate to confirm that a given space with an atlas is in fact a smooth manifold.

Example 1.18.

(i) Let $S^n = \{x = (x^1, \dots, x^{n+1}) \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$ denote the n-sphere, equipped with the subspace topology. Let $p_\pm = (0, \dots, \pm 1)$ denote the north $(+)$ and south $(-)$ pole. An atlas on S^n is given by the two charts

$$\sigma_+ : S^n \setminus \{p_+\} \to \mathbb{R}^n, \quad x \mapsto \frac{x}{1 - x^{n+1}},$$
$$\sigma_- : S^n \setminus \{p_-\} \to \mathbb{R}^n, \quad x \mapsto \frac{x}{1 + x^{n+1}}.$$

The firs thing to check is that the two chart cover S^n, which is simply observing that $S^n \setminus p_+ \cup S^n \setminus p_- = S^n$. Next, we need to check that all transition functions are smooth maps. We find that

$$\sigma_\pm^{-1} = \left(\frac{2x}{1 + \|x\|^2}, \pm \frac{\|x\|^2 - 1}{1 + \|x\|^2} \right).$$

Further calculation yields

$$\sigma_+ \circ \sigma_-^{-1} = \sigma_- \circ \sigma_+^{-1} : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}, \quad x \mapsto \frac{x}{\|x\|^2}.$$

Hence, the transition functions coincide and are given by the **inversion on the** $n-1$**-sphere** which is self-inverse and smooth.

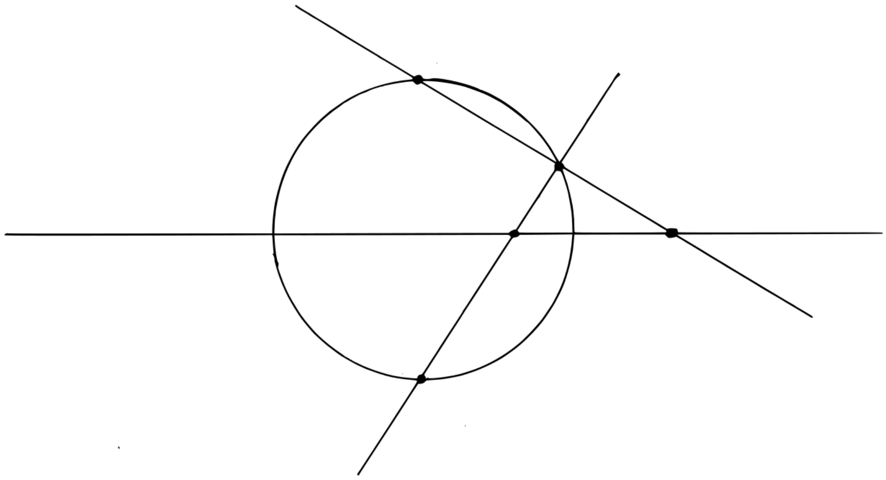


Figure 5: A sketch of the stereographic projection on S^1.

(ii) The n-dimensional **real projective space** $\mathbb{R}P^n$ is defined as the set of lines in $\mathbb{R}^{n+1}$. Formally, $\mathbb{R}P^n$ is the set of equivalence classes

$$\mathbb{R}P^n = \left\{ [x^1 : \ldots : x^{n+1}] \;\middle|\; x = (x^1, \ldots, x^{n+1}) \in \mathbb{R}^{n+1} \setminus \{0\} \right\},$$

where $[x] = [y]$ if $x = cy$ for some $c \in \mathbb{R} \setminus \{0\}$. This precisely means that the non-zero vectors x and y span the same line. One can check that $\mathbb{R}P^n$ equipped with the quotient topology induced by the canonical projection $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}P^n$, $x \mapsto [x]$, [3] is Hausdorff (draw a sketch!) and second countable. An atlas on $\mathbb{R}P^n$ is given by (φ_i, U_i), $1 \le i \le n$,

$$\begin{aligned} &\varphi_i : \pi\left(\mathbb{R}^{n+1} \setminus \{x^i = 0\}\right) \to \mathbb{R}^n, \\ &[x^1 : \ldots : x^{i-1} : x^i : x^{i+1} : \ldots : x^{n+1}] \mapsto \left(\frac{x^1}{x^i} : \ldots : \frac{x^{i-1}}{x^i} : \widehat{x^i} : \frac{x^{i+1}}{x^i} : \ldots : \frac{x^{n+1}}{x^i} \right), \end{aligned}$$

where " $\widehat{\ }$ " means that the element is supposed to be left out so that we end up with an n-vector. In order to check that the charts cover $\mathbb{R}P^n$ it suffices to check that $\bigcup 1 \le i \le n \mathbb{R}^{n+1} \setminus \{x^i \neq 0\} = \mathbb{R}^{n+1} \setminus \{0\}$. Next we need to check that all transition functions are smooth. Observe that the range of each φ_i is $\mathbb{R}^n$ for all $1 \le i \le n$ and that for all $i \neq j$

$$\varphi_j\left(\pi\left(\mathbb{R}^{n+1} \setminus \{x^i = 0\}\right) \cap \pi\left(\mathbb{R}^{n+1} \setminus \{x^j = 0\}\right)\right) = \begin{cases} \mathbb{R}^n \setminus \{x^i = 0\}, & i < j, \\ \mathbb{R}^n \setminus \{x^{i-1} = 0\}, & i > j. \end{cases}$$

Furthermore we have for all $1 \le j \le n$

$$\varphi_j^{-1}((x^1, \ldots, x^n)) = [x^1 : \ldots : x^{j-1} : 1 : x^j : \ldots : x^n].$$

Hence we obtain for all $i < j$

$$\varphi_i \circ \varphi_j^{-1} : \mathbb{R}^n \setminus \{x^i = 0\} \to \mathbb{R}^n \setminus \{x^{j-1} = 0\},$$

[3]The term "quotient topology" means that the open sets, in this case of $\mathbb{R}P^n$, are defined to be the images of open sets in the domain of the projection, in this case $\mathbb{R}^n \setminus \{0\}$. The notation of the elements in $\mathbb{R}P^n$ with the ":" is traditional.

$$(x^1, \dots, x^{n+1}) \mapsto \left(\frac{x^1}{x^i}, \dots, \widehat{\frac{x^i}{x^i}}, \dots, \frac{x^{j-1}}{x^i}, \frac{1}{x^i}, \frac{x^j}{x^i}, \dots, \frac{x^n}{x^i} \right),$$

and for $i > j$ we find a similar formula. We see that all transition functions are indeed smooth and conclude that $\mathbb{R}P^n$ with the provided atlas is indeed a smooth manifold. The local coordinate systems φ_i are called **inhomogeneous coordinates** on $\mathbb{R}P^n$.

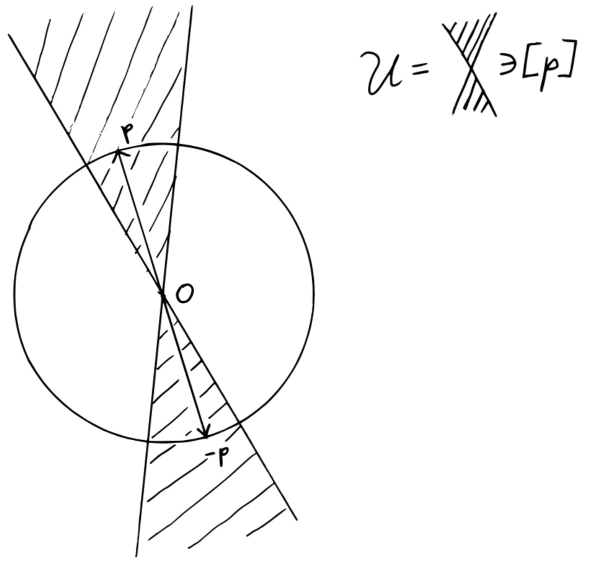


Figure 6: A subset U of $\mathbb{R}P^1$ is a set of lines through the origin $0 \in \mathbb{R}^2$.

(iii) Let $U \subset \mathbb{R}^n$ be open an $f : U \to \mathbb{R}$ a smooth map. Then the graph of f, $\mathrm{graph}(f) := \{(x, f(x)) \mid x \in U\} \subset \mathbb{R}^{n+1}$ is an n-dimensional smooth manifold with an atlas consisting of a single chart $\varphi : \mathrm{graph}(f) \to U$, $(x, f(x)) \mapsto x$.

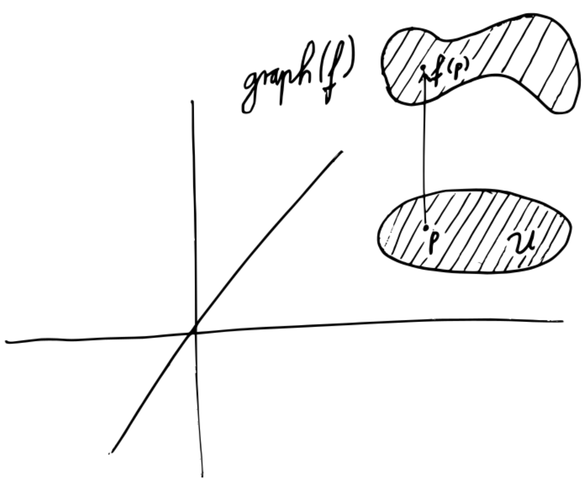


Figure 7: A sketch of the graph of a function $f : U \to \mathbb{R}$.

(iv) For a given smooth manifold M with atlas $\mathcal{A} = \{(\varphi_i, U_i) \mid i \in A\}$, any open subset $U \subset M$ equipped with the restriction of the atlas $\mathcal{A}$ to U, $\mathcal{A}|_U := \{(\varphi_i, U_i \cap U) \mid i \in A\}$, is a smooth manifold.

An other important class of smooth manifolds are so-called smooth submanifolds of a given smooth manifold. We will define that concept in full generality later, see Definition 1.57, but we already know from real analysis what a smooth submanifold of $\mathbb{R}^n$ is. Recall the following Theorem from real analysis.

Theorem 1.19. Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$, $(x, y) \mapsto f(x, y)$, be a smooth map and assume that $f(p) = 0$ for a point $p = (x_0, y_0) \in \mathbb{R}^n \times \mathbb{R}^m$ and that the Jacobi matrix of f with respect to y at p,

$$d_y f|_p = \begin{pmatrix} \frac{df_1}{dy^1}(p) & \dots & \frac{df_1}{dy^m}(p) \\ \vdots & \ddots & \vdots \\ \frac{df_m}{dy^1}(p) & \dots & \frac{df_m}{dy^m}(p) \end{pmatrix},$$

is invertible. Then there exists an open set $U \subset \mathbb{R}^n$ containing x_0 and an open set $V \subset \mathbb{R}^m$ containing y_0, such that there exists a unique smooth map $g : U \to V$ fulfilling

$$f(x, y) = 0,\ x \in U,\ y \in V \quad \Leftrightarrow \quad y = g(x).$$

In particular we have $g(x_0) = y_0$.

Definition 1.20. An $m < n$**-dimensional smooth submanifold of** $\mathbb{R}^n$ is a subset $M \subset \mathbb{R}^n$, such that for all $p \in M$ there exists an open set $U \subset \mathbb{R}^n$ containing p and a smooth map $f : U \to \mathbb{R}^{n-m}$ with Jacobi matrix of maximal rank $n - m$ for all points in U fulfilling

$$M \cap U = \{x \in U \mid f(x) = 0\}. \tag{1.4}$$

With the help of the implicit function theorem 1.19 it follows that locally up to re-ordering of coordinates on $\mathbb{R}^n$, any smooth $m < n$-dimensional submanifold M of $\mathbb{R}^n$ can be written as a graph of a smooth map $g : V \to \mathbb{R}^{n-m}$, $V \subset \mathbb{R}^m$ open. This in particular implies that, after possibly reordering the coordinates on $\mathbb{R}^n$, there exists locally near every point p in M a smooth invertible map with smooth inverse

$$F : U \to \mathbb{R}^n, \tag{1.5}$$

$p \in U$ and $U \subset \mathbb{R}^n$ open, such that

$$F|_{U \cap M} : (x^1, \dots, x^n) \mapsto (x^1, \dots, x^m, 0, \dots, 0). \tag{1.6}$$

We call F a locally defining function of M, which is motivated by $p \in U \cap M$ if and only if $u^{m+1}(F(p)) = \dots = u^n(F(p)) = 0$ after possibly shrinking U.

Exercise 1.21. Prove the above statements.

Now that we have recalled the definition of smooth submanifolds of $\mathbb{R}^n$, we need to ask ourselves if it is compatible with our general definition of smooth manifolds, see Definition 1.6.

Exercise 1.22. Show that smooth submanifolds of $\mathbb{R}^n$ are smooth manifolds. [Hint: Use that the inclusion map is smooth and construct new local coordinates on the ambient space $\mathbb{R}^n$ with the help of (1.6).]

Note that there are subsets of $\mathbb{R}^n$ which are not smooth submanifolds but can still be equipped with an atlas and, hence, are smooth manifolds:

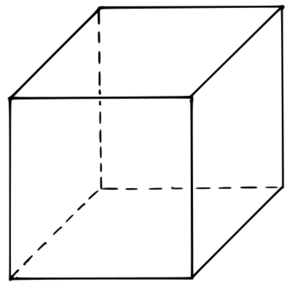


Figure 8: A cube.

Exercise 1.23. Show that the boundary of the unit cube $[0,1]^n \subset \mathbb{R}^n$ is not a smooth submanifold of $\mathbb{R}^n$ but can be equipped with a smooth atlas. Find an explicit example of such an atlas.

Yet another way to produce examples of smooth manifolds are products of smooth manifolds.

Lemma 1.24. Let M with atlas $\mathcal{A} = \{(\varphi_i, U_i) \mid i \in A\}$ be an m-dimensional smooth manifold and N with atlas $\mathcal{B} = \{(\psi_i, V_i) \mid i \in B\}$ be an n-dimensional smooth manifold. Then the Cartesian product of M and N, $M \times N$, equipped with the product topology and the product atlas $\mathcal{A} \times \mathcal{B} := \{(\varphi_i \times \psi_j, U_i \times V_j) \mid i \in A,\ j \in B\}$ is an $(m+n)$-dimensional smooth manifold.

Proof. This follows immediately from the definition of the product maps

$$\varphi_i \times \psi_j : U_i \times V_j \to \varphi_i(U_i) \times \psi_j(V_j) \subset \mathbb{R}^m \times \mathbb{R}^m \cong \mathbb{R}^{n+m}, \quad (p,q) \mapsto (\varphi_i(p), \psi_j(q)).$$

□

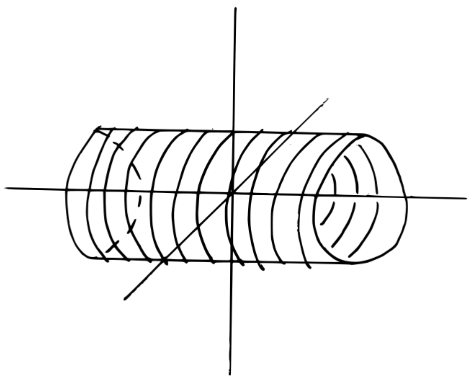


Figure 9: A rough sketch of the cylinder $S^1 \times (-1,1)$.

Exercise 1.25. Show that under the additional assumption that $\mathcal{A}$ and $\mathcal{B}$ are maximal in Lemma 1.24, the product atlas $\mathcal{A} \times \mathcal{B}$ is not necessarily maximal.

We now know what a smooth manifold is and we have seen some examples and counterexamples. Next we will define smooth maps between manifolds. In the language of category theory, these are the the homomorphism in the category of smooth manifolds.

Definition 1.26. Let M and N be smooth manifolds of dimension $m = \dim(M)$ and $n = \dim(N)$. A continuous map $f : M \to N$ is called **smooth** if for all charts (φ, U) of M, (ψ, V) of N, the map

$$\psi \circ f \circ \varphi^{-1} : \varphi(U \cap f^{-1}(V)) \to \psi(V), \tag{1.7}$$

is a smooth map between open sets in $\mathbb{R}^n$ and $\mathbb{R}^m$. By the term "f in local coordinates" we mean exactly the above formula (1.7) for a choice of charts (φ, U) and (ψ, V).

Definition 1.27. By $C^\infty(M)$ we denote the **$\mathbb{R}$-vector space of smooth $\mathbb{R}$-valued functions** on a smooth manifold M, that is all smooth maps in the sense of Definition 1.26 of the form

$$f : M \to \mathbb{R}.$$

If $U \subset M$ is open, U is by restriction of the atlas of M a smooth manifold itself. By $C^\infty(M)$ we denote all smooth function $f : U \to \mathbb{R}$

Exercise 1.28. Show that for $U \subset M$ open with $U \neq M$, the restriction map $C^\infty(M) \to C^\infty(U)$, $f \mapsto f|_U$, is in general not surjective (find a counterexample). Ask yourself what kind of difficulties might arise if one wanted to prove that the restriction map for any such U, M, is never surjective.

Example 1.29. An example of smooth functions on open subsets of a smooth manifold M are the local coordinates $x^i : U \to \mathbb{R}$ of a given chart (φ, U), cf. Definition 1.17. This is the reason why the x^i are also called **local coordinate functions**.

Definition 1.30.

(i) Let M, N be smooth manifolds. A smooth map $f : M \to N$ is called a **diffeomorphism** if it is invertible and its inverse is smooth.

(ii) Two manifolds M and N are called **diffeomorphic** if there exists a diffeomorphism $f : M \to N$.

Remark 1.31. There exist no two diffeomorphic smooth manifolds with different dimensions. This follows from the fact that every diffeomorphism is automatically a homeomorphism of the underlying topological spaces and, hence, locally a homeomorphism between open sets in $\mathbb{R}^m$ and $\mathbb{R}^n$. It follows from [Bro] that then $m = n$.

Exercise 1.32.

(i) Show that S^1 and $\mathbb{R}P^1$ are diffeomorphic.

(ii) Let M be a second countable Hausdorff topological space and let $\overline{\mathcal{A}}$ and $\overline{\mathcal{B}}$ be inequivalent maximal atlases on M. Prove that M equipped with $\overline{\mathcal{A}}$ is not diffeomorphic to M equipped with $\overline{\mathcal{B}}$.

1.2 Tangent spaces and differentials

So far we have introduced the "geometric" part of differential geometry in the sense that we have learned what the objects are that we will be studying, namely smooth manifolds. We have however not made sense of the "differential" part yet, which is what we will do next.

Remark 1.33. Recall the definition of tangent vectors in $\mathbb{R}^n$ that you know from real analysis. A tangent vector at a point $p \in \mathbb{R}^n$ is defined to be an equivalence class of smooth curves through p, $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$, $\gamma(0) = p$, where

$$[\gamma] = [\widetilde{\gamma}] \quad :\Leftrightarrow \quad \gamma'(0) = \widetilde{\gamma}'(0)$$

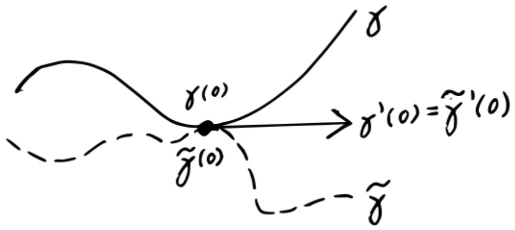


Figure 10: Two curves γ, $\widetilde{\gamma}$ with $\gamma(0) = \widetilde{\gamma}(0)$ that are in the same class.

So-defined tangent vectors act on locally near p defined smooth functions $f \in C^\infty(U)$, $p \in U$, $U \subset \mathbb{R}^n$ open, via

$$[\gamma]f := \left.\frac{d(f \circ \gamma)}{dt}\right|_{t=0} \tag{1.8}$$

which is precisely the directional derivative of f at p in the direction $\gamma'(0)$. Note that the value of $[\gamma]f$ depends, aside from $\gamma'(0)$, only on the values of f on an arbitrary small open neighbourhood of p in $\mathbb{R}^n$.

Furthermore recall that a tangent vector $[\gamma]$ at p is called tangential to a smooth $m < n$-dimensional submanifold M of $\mathbb{R}^n$ if for any locally defining function $F : M \cap U \to \mathbb{R}^n$, cf. equations (1.5) and (1.6), we have

$$dF_p \cdot \gamma'(0) \in \mathbb{R}^m \times \{0\}.$$

In the above equation, dF_p denotes the Jacobi matrix of F at p and the the statement of the equation just means that the last $n - m$ entries of $dF_p \cdot \gamma'(0)$ all vanish. Equivalently, $[\gamma]$ is tangential to M if it fulfils

$$df_p \cdot \gamma'(0) = 0$$

for a smooth map $f : U \to \mathbb{R}^{n-m}$ with Jacobi matrix of maximal rank with $M \cap U = \{x \in U \mid f(x) = 0\}$ for some open neighbourhood $U \subset \mathbb{R}^n$ of p.

The tangent space of $\mathbb{R}^n$ at $p \in \mathbb{R}^n$ is the collection of all tangent vectors at p and isomorphic to $\{p\} \times \mathbb{R}^n \cong \mathbb{R}^n$. The tangent space of $\mathbb{R}^n$ is the disjoint union of the tangent spaces at all points and is thereby given by $\mathbb{R}^{2n} \cong \mathbb{R}^n \times \mathbb{R}^n$. An element (p, v) in the tangent space has a base point p and a direction v which is the tangent vector.

We want to define tangent vectors and the tangent space for general smooth manifolds. The constructions should coincide (i.e. be isomorphic in some sense to be explained later, cf. Lemma **??**) with the above definition when considered for $\mathbb{R}^n$ viewed as a smooth manifold.

make Lemma: $J(\mathbb{R}, M) \cong TM$

Definition 1.34. Let M be a smooth manifold. A **tangent vector** v **at** $p \in M$ is a linear map

$$v : C^\infty(M) \to \mathbb{R},$$

that fulfils the Leibniz rule[4]

$$v(fg) = g(p)v(f) + f(p)v(g).$$

The set of tangent vectors at any fixed point $p \in M$ form a real vector space:

[4]Compare this to the Leibniz rule you know from real analysis!

Definition 1.35. The **tangent space at** $p \in M$

$$T_pM := \{v : C^\infty(M) \to \mathbb{R} \mid v \text{ tangent vector at p}\}$$

is the real vector space of all tangent vectors v at $p \in M$. This means that cv and $v + w$ are tangent vectors for all $c \in \mathbb{R}$, $v, w \in T_pM$, with

$$(cv)(f) = c \cdot v(f), \quad (v + w)(f) = v(f) + w(f),$$

for all $f \in C^\infty(M)$.

A central property of tangent vectors at $p \in M$ is that for any $f \in C^\infty$, $v(f)$ depends only on the values of f on an arbitrary small open neighbourhood of p. We will need the following tools.

Definition 1.36. A **smooth partition of unity** of a smooth manifold M is a set of smooth functions on M

$$\{f_i : M \to [0, 1] \mid i \in I\},$$

where I is an index set (e.g. $\mathbb{N}$ or $\mathbb{R}$), such that for all $x \in M$

$$\sum_{i \in I} f_i(x) = 1$$

A smooth partition of unity is called **locally finite** if

$$\{i \in I \mid f_i(x) \neq 0\}$$

is finite for all $x \in M$. If $\{U_i \subset M \mid i \in I\}$ is an open cover of M and $\operatorname{supp}(f_i) = \overline{\{x \in M \mid f_i(x) \neq 0\}}$ fulfils

$$\operatorname{supp}(f_i) \subset U_i$$

for all $i \in I$, then it the smooth partition of unity is called **subordinate to the open cover** $\{U_i \subset M \mid i \in I\}$.

Proposition 1.37. Let M be a smooth manifold and $\{U_i,\ i \in I\}$ an open covering of M. Then there exists a locally finite countable partition of unity on M subordinate to the open covering $\{U_i,\ i \in I\}$.

Proof. Exercise. [Hint: You might use your knowledge from real analysis and assume that the statement of this proposition is true for $M = \mathbb{R}^n$, $n \geq 1$. Also recall the existence of a countable atlas on any given manifold that is equivalent to the defining maximal atlas, see Exercise 1.12.] □

Definition 1.38. Let M be a smooth manifold and $U \subset M$ open. Let V be a subset of M with non-empty interior that is compactly embedded in U. Then a **bump function** with respect to the given data is a compactly supported smooth function $b \in C^\infty(M)$, such that

$$b|_{\overline{V}} \equiv 1, \quad \operatorname{supp}(b) \subset U. \tag{1.9}$$

Proposition 1.39. Let M, U, V be as in Definition 1.38 arbitrary but fixed. Then there exists a bump function b fulfilling (1.9).

Proof. We know from real analysis that this statement is true for $M = \mathbb{R}^n$. By using Proposition 1.37 it follows for arbitrary smooth manifolds as well. □

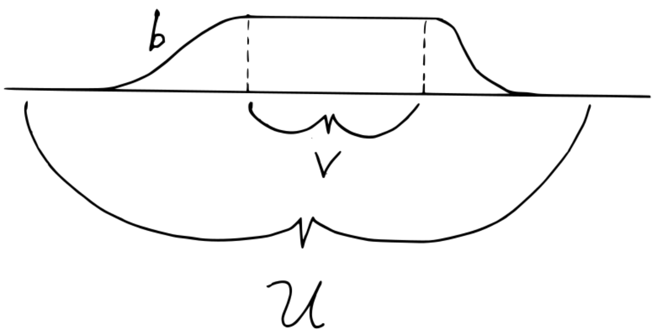


Figure 11: A bump function b w.r.t. V and U.

Exercise 1.40.

(i) Fill in the details of the proof of Proposition 1.39.

(ii) Let $U \subset M$ be any open subset of a smooth manifold M, $V \subset U$ a compactly embedded set with non-empty interior, and $b \in C^\infty(M)$ a bump function with respect to this data. Let $F : U \to \mathbb{R}^n$, $n \geq 1$, be a smooth map. Show that

$$bF : M \to \mathbb{R}^n, \quad (bF)(p) = b(p)F(p)\ \forall p \in U,\ (bF)(p) = 0\ \forall p \in M \setminus U$$

is smooth (the above globally on M defined map is called the **trivial extension** of $bF : U \to \mathbb{R}^n$ to M).

Now that we can use the existence of bump functions on smooth manifolds, we can continue our study of tangent vectors.

Proposition 1.41. Let $v \in T_pM$ be any tangent vector.

(i) Let $f, g \in C^\infty(M)$ and assume that for an open neighbourhood $U \subset M$ of $p \in M$, $f|_U = g|_U$. Then $v(f) = v(g)$.

(ii) Let $f \in C^\infty(M)$ be a smooth function that is locally constant near $p \in M$, meaning that there exists an open neighbourhood $U \subset M$ of p, such that $f|_U \equiv c$ for some $c \in \mathbb{R}$. Then $v(f) = 0$.

Proof. By the linearity of tangent vectors we have $v(f) = v(g)$ if and only if $v(f - g) = 0$. Thus, in order to prove (i) it suffices to show that if $v(f) = 0$ for some $f \in C^\infty(M)$ then f must already vanish near p, meaning that $f|_U \equiv 0$ for some open neighbourhood $U \subset M$ of p. Let $V \subset U$ be open and compactly embedded with $p \in V$ and fix any bump function $b \in C^\infty(M)$, such that

$$b|_V \equiv 1, \quad \text{supp}(b) \subset U.$$

Then $bf \equiv 0$ on M. By using the Leibniz rule for tangent vectors and $v(0) = 0$ by linearity we obtain

$$0 = v(0) = v(bf) = f(p)v(b) + b(p)v(f) = 0 + v(f).$$

Hence, $v(f) = 0$ as claimed.

We can now use (i) and find that for any locally constant function $f|_U \equiv c$ for some open neighbourhood $U \subset M$ of p, the value of $v(f)$ is the same as $v(c)$, where we view $c \in \mathbb{R}$ as the constant function on M with value c. We calculate

$$v(f) = v(c) = cv(1) = cv(1 \cdot 1) = c(1 \cdot v(1) + 1 \cdot v(1)) = 2cv(1) = 2v(f).$$

This shows that $v(f) = 0$. □

Remark 1.42. Proposition 1.41 shows that the action of tangent vectors at a point only depends on the local form of the functions near that point. This is sometimes phrased as "tangent vectors are local objects". This allows us to define the action of tangent vectors on functions that are only defined locally. Let $v \in T_pM$, $U \subset M$ an open neighbourhood of p, and let $f \in C^\infty(U)$. Since the the action of v on globally defined functions only depends on their behaviour near p, it is reasonable to define

$$v : f \mapsto v(bf),$$

for any bump function b with p contained in the interior of its support, so that $\text{supp}(b) \subset U$ and such that there exists $V \subset U$ compactly embedded with nonempty interior fulfilling $p \in V$, $V \subset \text{supp}(b)$, and $b|_{\overline{V}} \equiv 1$. Note that $v(bf)$ does not depend on the choice of such a bump function b. On the other hand, any open subset $U \subset M$ is a smooth manifold itself by restricting any atlas on M. This means that for any $p \in U$, the tangent space T_pU is well-defined. For any $\widetilde{v} \in T_pU$, we can define its action on $C^\infty(M)$ by

$$\widetilde{v}(f) := \widetilde{v}(f|_U).$$

By Proposition 1.41 we know that $\widetilde{v}(f)$ does in fact only depend on the behaviour of f on *any* open neighbourhood of p in U, which is then automatically an open neighbourhood of p in M. This means that we can canonically identify T_pM and T_pU for all $U \subset M$ open and all $p \in U$. From now on we will simply write $v(f)$ for $v \in T_pM$ and $f \in C^\infty(U)$ a locally defined smooth function with $p \in U$.

This motivates a slightly different definition of tangent vectors in T_pM as linear maps on the **germ of smooth functions at** $p \in M$

$$\mathcal{F}_p := \{f \in C^\infty(U) \mid p \in U,\ U \subset M \text{ open}\}/_\sim$$

where $f \sim g$ if and only if there exists $U \in M$ open and contained in the domain of definition of both f and g, such that $f|_U = g|_U$. The notion of a germ comes from sheaf theory. For an introduction see [Bre]. One can show that $\mathcal{F}_p$ is an $\mathbb{R}$-algebra and then define tangent vectors $v \in T_pM$ as linear maps

$$v : \mathcal{F}_p \to \mathbb{R}$$

fulfilling the Leibniz rule $v([f][g]) = g(p)v([f]) + f(p)v([g])$. This definition is equivalent to our definition with the same reasoning as for why T_pM and T_pU for $U \subset M$ open can be identified. This approach is used in [G].

While we have explained how a tangent vector should behave and have seen that it depends only on the local behaviour of functions, we do not yet have a convenient way to write down actual examples of tangent vectors. To do so we introduce specific tangent vectors that generalize partial derivatives we know from real analysis to smooth manifolds in any given local coordinate system.

Definition 1.43. Let $\varphi = (x^1, \ldots, x^n)$ be a local coordinate system on a smooth manifold M. The tangent vector $\frac{\partial}{\partial x^i}\Big|_p \in T_pM$ is defined as

$$\frac{\partial}{\partial x^i}\Big|_p (f) := \frac{\partial f}{\partial x^i}(p) := \frac{\partial (f \circ \varphi^{-1})}{\partial u^i}(\varphi(p))$$

for all $f \in C^\infty(M)$.

Lemma 1.44. $\frac{\partial}{\partial x^i}\Big|_p$ is a well-defined tangent vector.

Proof. Linearity follows from the fact that partial derivatives in $\mathbb{R}^n$ are linear with respect to scalar multiplication. For the Leibniz rule we recall that partial differentiation in $\mathbb{R}^n$ fulfils the Leibniz rule and calculate for any two $f, g \in C^\infty(M)$

$$\frac{\partial(f\cdot g)}{\partial x^i}(p) = \frac{\partial((f\cdot g)\circ\varphi^{-1})}{\partial u^i}(\varphi(p)) = \frac{\partial((f\circ\varphi^{-1})\cdot(g\circ\varphi^{-1}))}{\partial u^i}(\varphi(p))$$
$$= g(p)\frac{\partial(f\circ\varphi^{-1})}{\partial u^i}(\varphi(p)) + f(p)\frac{\partial(g\circ\varphi^{-1})}{\partial u^i}(\varphi(p)) = g(p)\frac{\partial f}{\partial x^i}(p) + f(p)\frac{\partial g}{\partial x^i}(p).$$

□

Important examples of $\left.\frac{\partial}{\partial x^i}\right|_p$ acting on smooth function are derivatives of coordinate functions.

Example 1.45. Let $\varphi = (x^1, \ldots, x^n)$ be a local coordinate system on a smooth manifold M covering $p \in M$. Then

$$\frac{\partial x^j}{\partial x^i}(p) = \delta_i^j$$

for all $1 \le i \le n$, $1 \le j \le n$. This follows from $(x^j \circ \varphi^{-1})(u^1, \ldots, u^n) = u^j$.

Using Definition 1.43 we can write down any tangent vector $v \in T_pM$ in a fixed local coordinate system (φ, U) with $p \in U$ as a linear combinations of the $\left.\frac{\partial}{\partial x^i}\right|_p$'s.

Proposition 1.46. For all $p \in M$ and any local chart $(\varphi = (x^1, \ldots, x^n), U)$ with $p \in U$, the set of tangent vectors $\left\{\left.\frac{\partial}{\partial x^i}\right|_p,\ 1 \le i \le n\right\}$ is basis of T_pM.

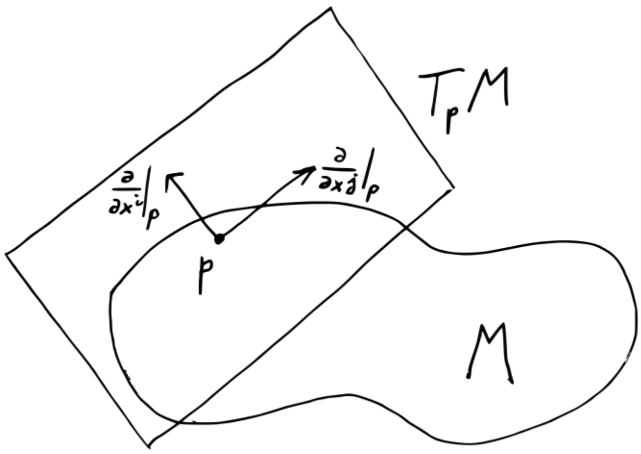


Figure 12: The tangent space T_pM at $p \in M$ is the linear span of the $\left.\frac{\partial}{\partial x^i}\right|_p$'s.

Proof. First we show that the set of $\left.\frac{\partial}{\partial x^i}\right|_p$'s is a linearly independent set of tangent vectors at p. Assume there exist $(c^1, \ldots, c^n) \in \mathbb{R}^n \neq 0$, such that

$$v_0 := \sum_{i=1}^n c^i \left.\frac{\partial}{\partial x^i}\right|_p$$

vanishes identically as a linear map $v_0 : C^\infty(M) \to \mathbb{R}$. There exists at least one $1 \le j \le n$, such that $c^j \neq 0$. But then

$$v_0(x^j) = c^j \neq 0$$

which is a contradiction to $v_0 = 0$.

Next we need to show that every tangent vector T_pM can be written as a linear combination of the $\frac{\partial}{\partial x^i}\big|_p$'s. Assume without loss of generality that $\varphi(U) = B_r(0)$ for some $r > 0$ with $\varphi(p) = 0$, that is the Euclidean unit ball with radius r. This can always be achieved by shrinking U and translating $\varphi(U)$ if necessary. For any smooth function g on $\varphi(U)$, it follows from the fundamental theorem of calculus[5] that with

$$g_i(q) := \int_0^1 \frac{\partial g}{\partial u^i}(tq)dt.$$

for all $q \in \varphi(U)$ we have

$$g = g(0) + \sum_{i=1}^{n} g_i u^i$$

on $\varphi(U)$. In particular, we obtain for any $f \in C^\infty(U)$ with $g = f \circ \varphi^{-1}$

$$f = g \circ \varphi = f(p) + \sum_{i=1}^{n} f_i x^i$$

where $f_i = g_i \circ \varphi$. By acting with the tangent vector $\frac{\partial}{\partial x^i}\big|_p$, $1 \leq i \leq n$, on both sides of the above equation we obtain[6]

$$f_i(p) = \frac{\partial f}{\partial x^i}(p).$$

Hence, we get using $x^i(p) = 0$ for $1 \leq i \leq n$ for $v \in T_pM$ fixed

$$v(f) = 0 + \sum_{i=1}^{n} \Big(v(f_i)x^i(p) + f_i(p)v(x^i) \Big) = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(p)v(x^i).$$

Since f was arbitrary this shows that the tangent vectors v and $\sum_{i=1}^{n} v(x^i)\, \frac{\partial}{\partial x^i}\big|_p$ coincide. Hence, v can be written as a linear combination of the proposed basis vectors. This finishes the proof. □

Corollary 1.47. The dimensions of a smooth manifold M and its tangent space T_pM coincide for all $p \in M$.

Example 1.48. As an example how the coordinate tangent vectors $\frac{\partial}{\partial x^i}\big|_p$ change for different coordinates consider the following example. Let $f \in C^\infty(M)$ be any smooth function and $(x^1, \ldots, x^n)$ be local coordinates covering $p \in M$. Then $(y^1, \ldots, y^n) := (2x^1, x^2, \ldots, x^n)$ are also local coordinates covering p. The vectors $\frac{\partial}{\partial x^i}\big|_p$ and $\frac{\partial}{\partial y^i}\big|_p$ coincide for $2 \leq i \leq n$, but for $i = 1$ we have

$$\frac{\partial}{\partial x^1}\bigg|_p = 2\, \frac{\partial}{\partial y^1}\bigg|_p.$$

See Figure 13 for a sketch of this example.

We now know the properties of tangent vectors and how they can be written locally, meaning that we can now properly calculate with them in fixed local coordinates. This allows us to define an analogue of the Jacobi matrix for smooth manifolds.

[5] If you do not see this, apply the fundamental theorem of calculus to $t \mapsto g(tq)$ for $q \in \varphi(U)$ fixed.

[6] Verifying this is a good exercise.

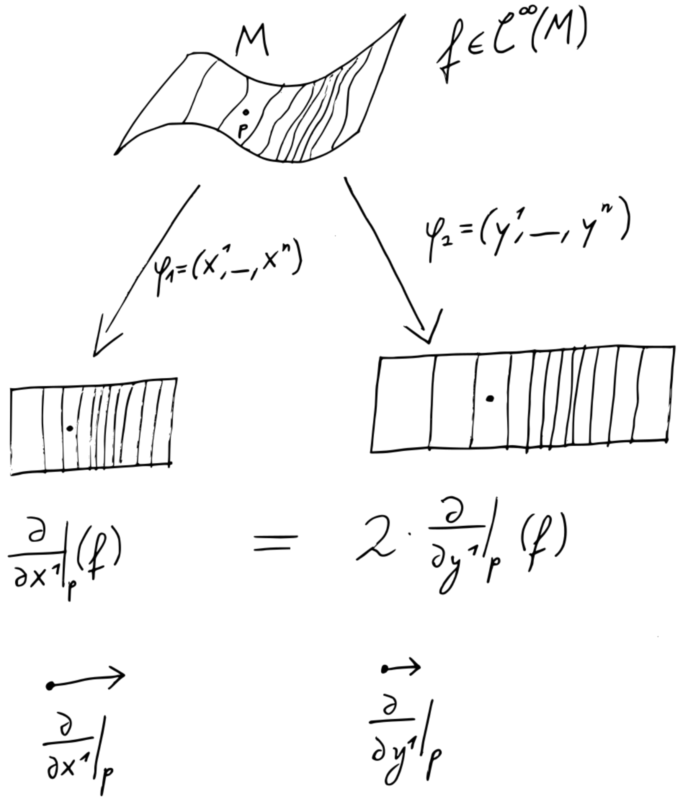


Figure 13: The lines in M, respectively the images of the charts, are supposed to be level sets of f.

Definition 1.49. Let M be a smooth manifold of dimension m and N be a smooth manifold of dimension n.

(i) The **differential at a point** $p \in M$ **of a smooth function** $f \in C^\infty(M)$ is defined as the linear map

$$df_p : T_pM \to \mathbb{R}, \quad v \mapsto v(f).$$

In a given local coordinate system $\varphi = (x^1, \dots, x^m)$ on M that covers p, df_p is of the form

$$df_p : \left.\frac{\partial}{\partial x^i}\right|_p \mapsto \frac{\partial f}{\partial x^i}(p).$$

(ii) The **differential at a point** $p \in M$ **of a smooth map** $F : M \to N$ in given local coordinate systems $\varphi = (x^1, \dots, x^m)$ on M and $\psi = (y^1, \dots, y^n)$ on N covering $p \in M$ and $F(p) \in N$, respectively, is defined as the linear map

$$dF_p : T_pM \to T_{F(p)}N, \quad \left.\frac{\partial}{\partial x^i}\right|_p \mapsto \sum_{j=1}^n \frac{\partial F^j}{\partial x^i}(p) \left.\frac{\partial}{\partial y^j}\right|_{F(p)},$$

where we have used the notation

$$F^j := y^j \circ F.$$

The **rank** of F at p is the rank of the linear map $dF_p : T_pM \to T_{F(p)}N$, which coincides with the rank of the **Jacobi matrix of** F **at** p in the local coordinate systems φ, ψ,

$$\left(\frac{\partial F^j}{\partial x^i}(p)\right)_{ji} \in \mathrm{Mat}(n \times m, \mathbb{R}).$$

In the above equation, j is the row and i is the column of the matrix.

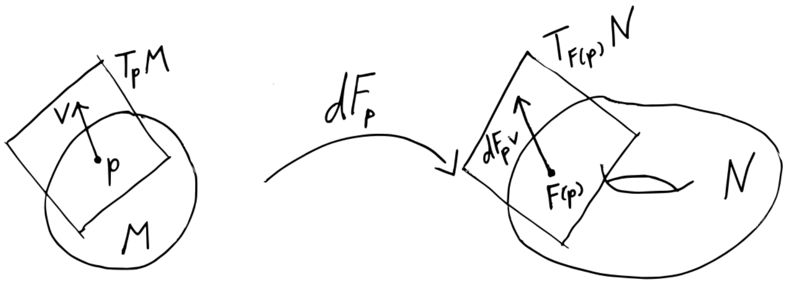


Figure 14: Sketch of the differential at $p \in M$ of a smooth map $F : M \to N$.

Example 1.50. Let $\varphi = (x^1, \dots, x^n)$ be a local coordinate system on a smooth manifold M covering $p \in M$. Then $d\varphi_p : T_pM \to T_{\varphi(p)}\mathbb{R}^n \cong \mathbb{R}^n$ is of the form

$$d\varphi_p = (dx^1, \dots, dx^n)_p = (dx^1_p, \dots, dx^n_p), \quad dx^j_p \left(\frac{\partial}{\partial x^i}\bigg|_p \right) = \delta^j_i.$$

We will usually omit the base point and simply write $dx^i := dx^i_p$ if it is clear from either the context or the tangent vector's base point that dx^i acts on.

Remark 1.51. Note that Definition 1.49 (i) is a special case of (ii) (using the canonical coordinate u^1 on $\mathbb{R}$).

Similar to real analysis, the differential of smooth maps between smooth manifolds fulfils the following chain rule.

Lemma 1.52. Let M, N, P be smooth manifolds and $F : M \to N$, $G : N \to P$, smooth maps. Then

$$d(G \circ F)_p = dG_{F(p)} \circ dF_p$$

for all $p \in M$.

Proof. For any $v \in T_pM$ and $f \in C^\infty(P)$ we have

$$d(G \circ F)_p(v)(f) = v(f \circ G \circ F) = dF_p(v)(f \circ G) = dG_{F(p)}(dF_p(v))(f).$$

□

Definition 1.53.

(i) A smooth map between smooth manifolds $F : M \to N$ is called an **immersion** if $dF_p : T_pM \to T_{F(p)}N$ is injective for all $p \in M$.

(ii) $F : M \to N$ is called a **submersion** if $dF_p : T_pM \to T_{F(p)}N$ is surjective for all $p \in M$.

(iii) An immersion $F : M \to N$ is called an **embedding** if F is injective and an homeomorphism onto its image $F(M) \subset N$ equipped with the subspace topology.

(iv) A smooth map $F : M \to N$ between smooth manifolds of the same dimension is called a **local diffeomorphism** if for all $p \in M$ there exists an open neighbourhood of p, $U \subset M$, such that $F|_U : U \to N$ is a diffeomorphism onto its image.

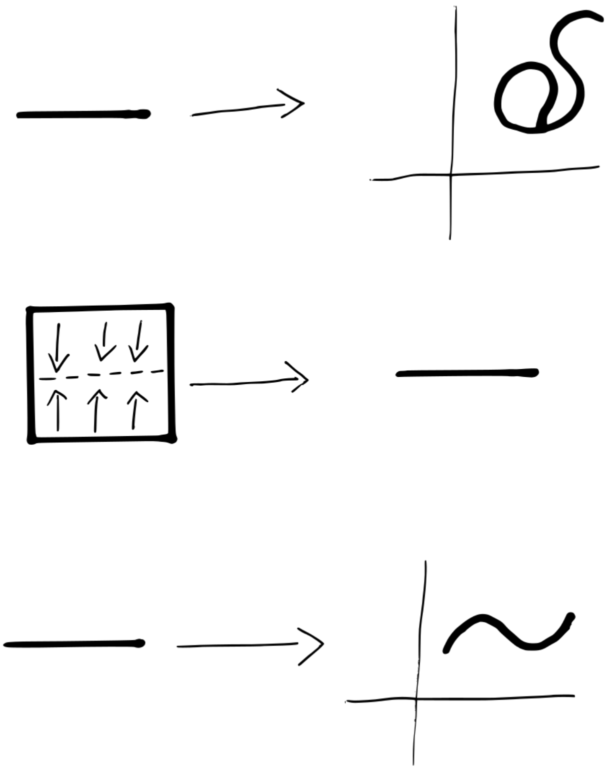


Figure 15: An immersion, an embedding, and a submersion. Which is which?

Suppose that we are given a smooth map $F : M \to N$, $\dim(M) = \dim(N)$, and want to check if it is a local diffeomorphism. At first this sounds fairly complicated, but luckily we can use the following result.

Theorem 1.54. Let $F : M \to N$ be a smooth map between two manifolds of the same dimension n and let $p \in M$ be arbitrary. Then $dF_p : T_pM \to T_{F(p)}N$ is a linear isomorphism if and only if there exists an open neighbourhood $U \subset M$ of p, such that $F|_U$ is a diffeomorphism onto its image.

Proof. Let (φ, U) and (ψ, V) be local charts covering $p \in M$ and $F(p) \in N$, respectively. Observe that, by definition, dF_p is a linear isomorphism if any only if its Jacobi matrix in the given local coordinates is invertible. On the other hand, there exists an open neighbourhood $U \subset M$ of p, such that $F|_U$ is a diffeomorphism onto its image if and only if there exist open sets $U', V' \subset \mathbb{R}^n$ with $\varphi(p) \in U'$, $\psi(F(p)) \in V'$, such that

$$\psi \circ F \circ \varphi^{-1} : U' \to V'$$

is a diffeomorphism. We can without loss of generality assume that $\varphi(U) = U'$ and $\psi(V) = V'$. Hence, the "$\Rightarrow$"-direction of the statement of this theorem follows from the inverse function theorem[7]. The "$\Leftarrow$"-direction follows from the fact that invertible smooth maps with smooth inverse in the real analysis setting have pointwise invertible Jacobi matrix. □

Corollary 1.55. $F : M \to N$ is a local diffeomorphism if and only if dF_p is a linear isomorphism for all $p \in M$.

[7]Which, in turn, follows from the implicit function theorem. Note, however, that one usually proves the implicit function theorem using the inverse function theorem, see e.g. [**?**, R]

Exercise 1.56.

(i) Find explicit examples of an immersion that is not injective and an injective immersion that is not an embedding.

(ii) Show that for all $n \in \mathbb{N}$, the map

$$\pi : S^n \to \mathbb{R}P^n, \quad (x^1, \dots, x^n) \mapsto [x^1 : \dots : x^n]$$

is a local diffeomorphism but not a diffeomorphism.

1.3 Submanifolds

We already know what a smooth submanifold of $\mathbb{R}^n$ is. Using Definition 1.53 we can now define what a smooth submanifold in our more general setting should be.

Definition 1.57. Let N be an n-dimensional and M be an m-dimensional smooth manifold. Let further $F : M \to N$ be a smooth map.

(i) $F(M) \subset N$ is called an **embedded smooth submanifold** if F is an embedding.

(ii) If F is the inclusion map $\iota : M \hookrightarrow N$, we will say that $M \subset N$ is a **smooth submanifold** if the inclusion is an embedding.

(iii) If $M \subset N$ is a smooth submanifold, the number $\dim(N) - \dim(M)$ is called the **codimension** of M in N. Smooth submanifolds of codimension 1 are called **hypersurfaces**.

We will be mainly concerned with smooth submanifolds that are given as subsets of the ambient manifold. The first thing one should ask is how to obtain a the structure of a smooth manifold on a submanifold and if it coincides with the initial manifold structure.

Proposition 1.58. Let $M \subset N$, $\dim(M) = m < n = \dim(N)$, be a smooth submanifold and let $p \in M$ be arbitrary. Then there exists a chart[8] $(\varphi = (x^1, \dots, x^n), U)$ on N, such that $U \cap M$ is an open neighbourhood of p in M and

$$x^{m+1}(q) = \dots = x^n(q) = 0$$

for all $q \in U \cap M$. The first m entries in φ are a local coordinate system on M near p.

Proof. Fix $p \in M \subset N$ and choose local coordinates $(x^1, \dots, x^n)$ on N and $(y^1, \dots, y^m)$ on M covering p. Since M is a submanifold of N, the differential of the inclusion map $\iota : M \to N$ at p, $d\iota_p$, is injective and its Jacobi matrix

$$\left(\frac{\partial x^i}{\partial y^j}(p) \right)_{ij} \in \mathrm{Mat}(n \times m, \mathbb{R})$$

has rank m. After reordering the x^i-coordinate functions, we can assume without loss of generality that the first m rows are linearly independent. By the implicit function theorem that means that the first m coordinates on N form, by restriction, a coordinate system on an open set $V \subset M$ containing p. Furthermore, after possibly shrinking V, we have again by the implicit function theorem that $(q^1, \dots, q^n) \in \iota(V) \subset N$ if and only if $x^k(q) = f^k(x^1(q), \dots, x^m(q))$ for uniquely defined functions $f^k : (x^1, \dots, x^m)(V) \to \mathbb{R}$ for all $m + 1 \leq k \leq n$. Choose an open

[8] This means that this chart is compatible with the given maximal atlas on N.

subset $U \subset N$, so that the local coordinates $(x^1, \dots, x^n)$ are defined on V, $U \cap N = V$, and define on U smooth functions

$$F^k := x^k - f^k(x^1, \dots, x^m), \quad m+1 \leq k \leq n.$$

In the last step we will define new coordinates on N fulfilling the statement of this proposition as follows. Define

$$\varphi : U \to \mathbb{R}^n, \quad \varphi = (x^1, \dots, x^m, F^{m+1}, \dots, F^n).$$

The Jacobi matrix of φ at p with respect to the coordinates $(x^1, \dots, x^n)$ is of the form

$$\begin{pmatrix} \mathrm{id}_{\mathbb{R}^m} & 0 \\ A & \mathrm{id}_{\mathbb{R}^{n-m}} \end{pmatrix}$$

for some real-valued matrix $A \in \mathrm{Mat}((n-m) \times m, \mathbb{R})$. The above Jacobi matrix is in particular invertible, showing that φ is a local diffeomorphism. Furthermore

$$\varphi(U \cap M) = \varphi(V) = (x^1, \dots, x^m, 0, \dots, 0).$$

Hence, the first m entries restriction of φ to V is a local coordinate system on M near p and fulfils the claims of this proposition. □

Definition 1.59. Local coordinates as in Proposition 1.58 for a submanifold $M \subset N$ near a given point $p \in M$ are called **adapted coordinates**.

An important consequence of Proposition 1.58 is that the smooth structure of a manifold that can be realized as a smooth submanifold coincides with the smooth structure obtained by adapted coordinates:

Corollary 1.60. Any smooth manifold M that can be realized as a submanifold of some ambient manifold N is diffeomorphic to M, viewed as a topological subspace of N, equipped with any atlas consisting only of adapted coordinates.

Note that adapted coordinates relate the definition of smooth submanifolds of $\mathbb{R}^n$ to the more general Definition 1.57, cf. equation (1.6). Furthermore observe that Corollary 1.60 also means that if we can cover a topological subspace[9] of M by adapted coordinates it will automatically be a submanifold of M. For a more detailed explanation of the latter see [L2, Thm. 5.8]. One way to construct explicit examples of submanifolds is via pre-images of regular values of smooth maps between smooth manifolds.

Definition 1.61. Let M and N be smooth manifolds and let $F : M \to N$ be a smooth map. A point $p \in M$ is called **regular point** of F if $dF_p : T_pM \to T_{F(p)}N$ is surjective. Any point $q \in N$, such that $F^{-1}(q) \subset M$ consists only of regular points, is called **regular value** of F. Points in M that are not regular points of F are called **critical points** of F, and points in N such that the pre-image under F in M contains at least one critical point of F are called **critical values** of F.

Note that for a smooth map $F : M \to N$ to have regular values it is a necessary condition that $\dim(M) \geq \dim(N)$.

Proposition 1.62. Let M and N be smooth manifolds with $\dim(M) = m \geq n = \dim(N)$. Let $F : M \to N$ be smooth and let $q \in N$ be a regular value of F. Then the level set

$$F^{-1}(q) \subset M$$

is an $(m-n)$-dimensional smooth submanifold of M. The structure of a smooth manifold on $F^{-1}(q)$ is uniquely determined by requiring that the inclusion is smooth.

[9] Careful, needs to be second countable and Hausdorff.

For the proof of the above definition we need the following definition and two theorems, cf. [L2, Thm. 4.12, Thm. 5.12].

Definition 1.63. Let $F : M \to N$ be a smooth maps between smooth manifolds and let (φ, U) and (ψ, V) be local charts of M and N, respectively, such that $F(U) \subset V$. Let further $\dim(M) = m$ and $\dim(N) = n$. The **coordinate representation of** F in the local coordinate systems φ and ψ is defined to be the smooth map

$$\widehat{F} : \varphi(U) \to \varphi(V), \quad \widehat{F}(u^1, \dots, u^m) := (\psi \circ F \circ \varphi^{-1})(u^1, \dots, u^m).$$

Theorem 1.64. Let M be an m-dimensional and N be an n-dimensional smooth manifold. Let $F : M \to N$ be a smooth map of constant rank r. Then for each $p \in M$ there exist local charts (φ, U) of M with $p \in U$ and (ψ, V) of N with $F(p) \in V$, such that $F(U) \subset V$ and that the coordinate representation of F is of the form

$$\widehat{F}(u^1, \dots, u^r, u^{r+1}, \dots, u^m) = (u^1, \dots, u^r, 0, \dots, 0).$$

Proof. For a detailed proof see [L2, Thm. 4.12]. The case $r = 0$ is left as an exercise. Assume that $r \geq 1$. The proof works as follows. Firstly, for $p \in M$ fixed we choose any local coordinates M covering p and of N covering $F(p)$. Since the statement of this theorem is local, by witching to local coordinates we find that in order to prove it it suffices to consider the special case $M \subset \mathbb{R}^m$ open and $N \subset \mathbb{R}^n$ open. This shows that this theorem is equivalent to the rank theorem known from real analysis, see (in a slightly different formulation) [R, Thm. 9.32]. □

Theorem 1.65. Let M and N be smooth manifolds and $F : M \to N$ smooth and of constant rank r. Each level set $F^{-1}(q) \subset M$, $q \in N$, is a smooth submanifold of codimension r in M.

Proof. Let $q \in N$ and $p \in F^{-1}(q)$ be fixed. Using Theorem 1.64 we chose charts $(\varphi = (x^1, \dots, x^m), U)$ of M with $p \in U$ and (ψ, V) of N with $q \in V$ fulfilling $\varphi(p) = 0$ and $\psi(q) = 0$, such that the coordinate representation $\widehat{F}$ of F is of the form

$$\widehat{F} : \varphi(U) \to \varphi(V), \quad \widehat{F}(u^1, \dots, u^r, u^{r+1}, \dots, u^m) = (u^1, \dots, u^r, 0, \dots, 0).$$

Then $(\psi \circ F)(x^1, \dots, x^m) = (x^1, \dots, x^r, 0, \dots, 0)$ and, hence,

$$F^{-1}(q) \cap U = \{p \in U \mid x^1(p) = \ldots = x^r(p) = 0\}.$$

We see that such a coordinate choice (up to reordering) of M yields adapted coordinates on $F^{-1}(q) \cap U$. Since the rank of F is constant we can cover $F^{-1}(q)$ with such adapted coordinates and obtain that it is, in fact, a smooth submanifold of M. □

Lastly we will need the following fact.

Proposition 1.66. Let M and N be smooth manifolds and $F : M \to N$ any smooth map. Suppose that $p \in M$ is a regular point of F. Then there exists an open neighbourhood $U \subset M$ of p, such that all points in U are regular points of F. In particular this means that the set of regular points of F is open in M.

Proof. Exercise. [Hint: Use local coordinates to reduce the proof to the case $M \subset \mathbb{R}^m$ open and $N \subset \mathbb{R}^n$ open.] □

Proof of Proposition 1.62. Let $F : M \to N$ be smooth and $q \in N$ a regular value of F. By Proposition 1.66 the set

$$\mathrm{reg}(F) := \{p \in M \mid p \text{ regular point of } F\}$$

is open in M and thereby a smooth submanifold of M. We further have $F^{-1}(q) \subset \operatorname{reg}(F)$. The restriction of F to $\operatorname{reg}(F)$,

$$F|_{\operatorname{reg}(F)} : \operatorname{reg}(F) \to N$$

is by Definitions 1.53 and 1.61 a submersion and thereby of constant rank equal to $\dim(N)$. Using Theorem 1.65 it follows that $F^{-1}(q) \subset \operatorname{reg}(F)$ is a smooth submanifold. Since the composition of the inclusions $F^{-1}(q) \subset \operatorname{reg}(F)$ and $\operatorname{reg}(F) \subset M$ is still the inclusion and thereby in particular still a smooth embedding it follows that $F^{-1}(q) \subset M$ is a smooth submanifold. Since $\operatorname{reg}(F) \subset M$ is open it follows with Theorem 1.65 that $\dim(F^{-1}(q)) = m - n$. □

1.4 Vector bundles and sections

Up to this point, we know what tangent vectors at a specific given point are. The next step is to study vector fields, that is maps that assign to points in a manifold tangent vectors in their respective tangent spaces. In the general setting of smooth manifold these objects are more involved than in the case we know from real analysis.

Remark 1.67. Recall that a smooth vector field on $\mathbb{R}^n$ is a smooth vector valued function

$$X : \mathbb{R}^n \to \mathbb{R}^n, \quad p \mapsto X_p,$$

and we think of points $(p, X_p) \in \mathbb{R}^n \times \mathbb{R}^n$ as tangent vectors X_p with basepoint p. An example is the position vector field $X : p \mapsto p$. Observe that vector fields on $\mathbb{R}^n$, similar to tangent vectors, act on functions via

$$X(f) : \mathbb{R}^n \to \mathbb{R}, \quad p \mapsto [\gamma]f,$$

where $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ is any smooth curve, such that

$$\gamma(0) = p, \quad \gamma'(0) = X(p),$$

cf. equation (1.8). Note that for any smooth vector field X on $\mathbb{R}^n$ and any $f \in C^\infty(\mathbb{R}^n)$, $X(f) \in C^\infty(\mathbb{R}^n)$. One might also write $X(f) = df(X) : p \mapsto df_p(X_p)$.

Definition 1.68. A **vector bundle** $E \to M$ **of rank** $k \in \mathbb{N}$ over a smooth manifold M is a smooth manifold E together with a smooth **projection map** $\pi : E \to M$, such that

(i) the **fibre** $E_p := \pi^{-1}(p)$ is an k-dimensional real vector space for all $p \in M$,

(ii) for all $p \in M$ there exists an open neighbourhood $U \subset M$ of p and a diffeomorphism $\psi : \pi^{-1}(U) \to U \times \mathbb{R}^k$, such that $\psi|_{E_q} : E_q \to q \times \mathbb{R}^k \cong \mathbb{R}^k$ is a linear isomorphism for all $q \in U$ and the diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\quad\psi\quad} & U \times \mathbb{R}^k \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\mathrm{pr}_U} & \\ & U & \end{array}$$

commutes. The map pr_U denotes the canonical projection onto the first factor.

E is called the **total space**, M is called the **basis**, and the map ψ is called a **local trivialization** of the vector bundle $E \to M$.

Vector bundles provide the setting for an analogue to vector valued functions on smooth manifolds.

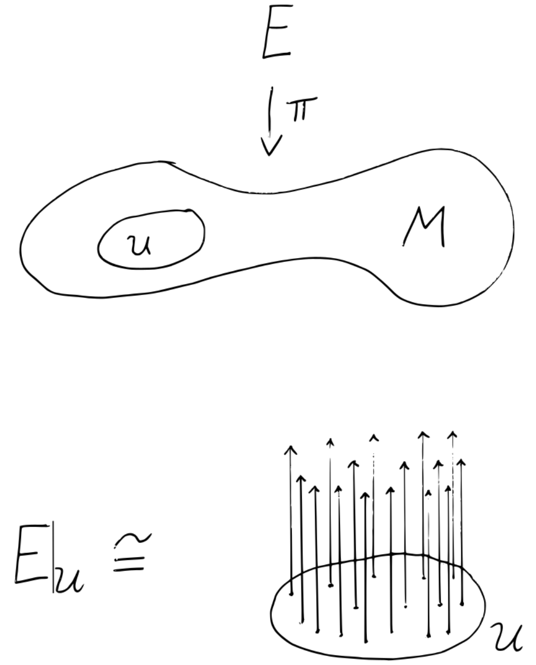


Figure 16: Locally, $E|_U \cong U \times \mathbb{R}^k$.

Definition 1.69. Let $E \to M$ be a vector bundle. A **local section in** $E \to M$ is a smooth map

$$s : U \to E$$

with $U \subset M$ open, such that $\pi \circ s = \mathrm{id}_U$. This precisely means that $s(p) \in E_p$ for all $p \in U$. If $U = M$, s is called a **(global) section**. The set of local sections in $E \to M$ on $U \subset M$ is denoted by $\Gamma(E|_U)$ and the set of global sections by $\Gamma(E)$, where $E|_U$ denotes the vector bundle $\pi^{-1}(U) \to U$. The **support** of a section (or, analogously, local section) in a vector bundle $s \in \Gamma(E)$ is defined to be the set

$$\mathrm{supp}(s) := \overline{\{p \in M \mid s(p) \neq 0\}}.$$

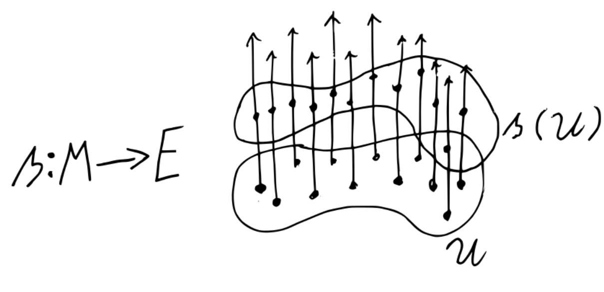


Figure 17: A sketch of a section.

Exercise 1.70.

(i) Show that $\Gamma(E)$ is a $C^\infty(M)$-module. Also show that $\Gamma(E|_U)$ is a $C^\infty(U)$-module for all $U \subset M$ open.

(ii) Show that for $k > 0$ the restriction map $\Gamma(E) \to \Gamma(E|_U)$ for $U \subset M$ open and precompact, such that the boundary of U, ∂U, is nonempty and a smooth hypersurface in M, is not surjective.

Definition 1.71. Let $\psi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ and $\phi : \pi^{-1}(V) \to V \times \mathbb{R}^k$ be two local trivializations of a vector bundle $E \to M$. Assume that $U \cap V \neq \emptyset$. Then the smooth map

$$\psi \circ \phi^{-1} : (U \cap V) \times \mathbb{R}^k \to (U \cap V) \times \mathbb{R}^k$$

is called **transition function**[10]. For $p \in M$ fixed, $(\psi \circ \phi^{-1})(p, \cdot)$ is called **transition function at** p.

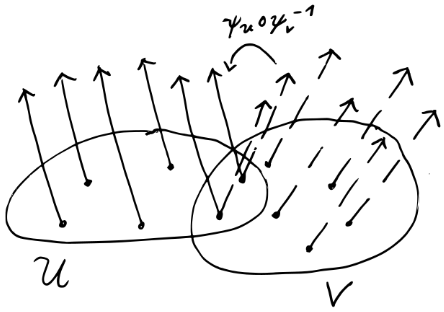


Figure 18: How a transition function w.r.t. (ψ_U, U) and (ψ_V, V) from the overlap of $V \times \mathbb{R}^k$ and $U \times \mathbb{R}^k$ to itself can be imagined.

Lemma 1.72. Transition functions $\psi \circ \phi^{-1}$ as in Definition 1.71 are of the form

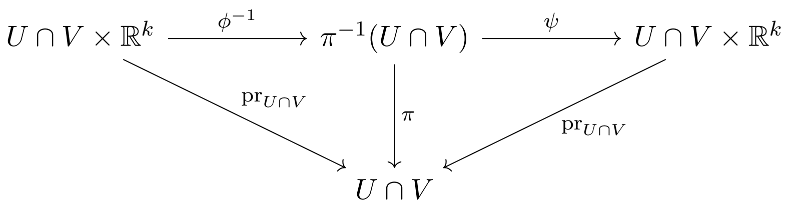
$$\psi \circ \phi^{-1} : (p, v) \mapsto (p, A(p)v), \quad A(p) \in \mathrm{GL}(k),$$

for all $p \in U \cap V$, $v \in \mathbb{R}^n$. The map

$$A : U \cap V \to \mathrm{GL}(k), \quad p \mapsto A(p),$$

is smooth.

Proof. The diagram



commutes and, hence, it is clear that $\psi \circ \phi^{-1}$ sends (p, v) to $(p, A(p, v))$ for some smooth function $A : U \cap V \times \mathbb{R}^k \to \mathbb{R}^k$. Smoothness follows from the diffeomorphism property of ϕ and ψ. We need to show that for p fixed, $A(p, \cdot) : \mathbb{R}^k \to \mathbb{R}^k$ is an invertible linear map. This follows from the fact that fibrewise ϕ and ψ are linear isomorphisms. $\square$

An important tool to construct vector bundles is from, heuristically speaking, a given set transition functions.

[10] Note that a change of coordinates in a smooth manifold is also called a transition function. Always make sure to clarify which kind of transition functions you are dealing with.

Proposition 1.73 ("Vector bundle chart lemma"). Let M be a smooth manifold and assume that for every $p \in M$, E_p is a real vector space of fixed dimension k. Define a set

$$E := \bigsqcup_{p \in M} E_p$$

together with a map $\pi : E \to M$, $\pi(v) = p$ for all $v \in E_p$ and all $p \in M$. Assume that $\{U_i,\ i \in I\}$ is an open cover of M and for each $i \in I$,

$$\phi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^k$$

is a bijection with the property that the restriction $\phi_i : E_p \to \{p\} \times \mathbb{R}^k \cong \mathbb{R}^k$ is a linear isomorphism for all $p \in M$. Further assume that for all $i, j \in I$ with $U_i \cap U_j \neq \emptyset$ there exists a smooth map $\tau_{ij} : U_i \cap U_j \to \mathrm{GL}(k)$, such that $\phi_i \circ \phi_j^{-1} : (U_i \cap U_j) \times \mathbb{R}^k \to (U_i \cap U_j) \times \mathbb{R}^k$ is of the form

$$\phi_i \circ \phi_j^{-1}(p, v) = (p, \tau_{ij}(p)v).$$

Then there exists a unique topology and maximal atlas on E, such that $\pi : E \to M$ is a vector bundle of rank k and the ϕ_i, $i \in I$, are local trivializations.

Proof. The proof follows [L2, Lem. 10.6]. Without loss of generality assume that we can find an atlas $\{(\varphi_i, U_i) \mid i \in I\}$ on M. This can always be achieved by shrinking the U_i if necessary and, on possible new overlaps $U_i \cap U_j$, set $\tau_{ij} \equiv \mathrm{id}_{\mathbb{R}^k}$. Now we can explicitly construct an atlas on the total space E. Define for $i \in I$

$$\psi_i : \pi^{-1}(U_i) \to \varphi_i(U_i) \times \mathbb{R}^k, \quad v \mapsto (\varphi_i \times \mathrm{id}_{\mathbb{R}^k})(\phi_i(v)).$$

In order for $\{(\psi_i, \pi^{-1}(U_i)) \mid i \in I\}$ to be a smooth atlas on E, we need to show that the transition functions (as in transition functions of a smooth atlas, cf. Definition 1.4) are smooth. We check that

$$\psi_i(\pi^{-1}(U_i) \cap \pi^{-1}(U_j)) = \varphi_i(U_i \cap U_j) \times \mathbb{R}^k$$

for all $i, j \in I$, and we find

$$\psi_i \circ \psi_j^{-1} = (\varphi_i \times \mathrm{id}_{\mathbb{R}^k}) \circ (\phi_i \circ \phi_j^{-1}) \circ (\varphi_j^{-1} \times \mathrm{id}_{\mathbb{R}^k}) : \varphi_j(U_i \cap U_j) \times \mathbb{R}^k \to \varphi_i(U_i \cap U_j) \times \mathbb{R}^k.$$

Since, by assumption, $\tau_{ij}(p)$ is invertible and depends smoothly on $p \in U_i \cap U_j$, $\phi_i \circ \phi_j^{-1}$ is a diffeomorphism for all $i, j \in I$ such that $U_i \cap U_j \neq \emptyset$. Since the φ_i form a smooth atlas on M, each $\varphi_i \times \mathrm{id}_{\mathbb{R}^k}$ is a diffeomorphism. Hence, $\psi_i \circ \psi_j^{-1} : \varphi_j(U_i \cap U_j) \times \mathbb{R}^k \to \varphi_i(U_i \cap U_j) \times \mathbb{R}^k$ is also a diffeomorphism for all $i, j \in I$ such that $U_i \cap U_j \neq \emptyset$. By defining the open sets on E as the preimages of open sets under ψ_i, $i \in I$, we have that it is second countable and Hausdorff by the assumption that M is a smooth manifold (and $\mathbb{R}^k$ is, of course, second countable and Hausdorff as well). Equipped with the so-defined topology, $\mathcal{B} := \{(\psi_i, \pi^{-1}(U_i)) \mid i \in I\}$ is a smooth atlas on the total space E. Then all the maps ϕ_i, $i \in I$, are automatically smooth and, since $\phi_i : E_p \to \{p\} \times \mathbb{R}^k$ is a linear isomorphism by assumption, form a covering of local trivializations which turns $E \to M$ into a vector bundle of rank k. The uniqueness of the smooth manifold structure on E now follows from the assumption that all ϕ_i are diffeomorphisms onto their image and, thus, every smooth atlas on E with that property must, by construction, be a refinement of $\mathcal{B}$ and thus be contained in the same maximal smooth atlas as $\mathcal{B}$. □

Now we have all the tools at hand that we need to define the tangent bundle of a smooth manifold.

Definition 1.74. Let M be an n-dimensional smooth manifold. The **tangent bundle**[11] $TM \to M$ of M is a vector bundle of rank n with total space $TM := \bigsqcup_{p \in M} T_pM$ and projection $\pi(v) = p$ for all $v \in T_pM$.

At this point, however, we still need to explain the structure of a smooth manifold on the total space of the tangent bundle TM and we need to show that it actually is a vector bundle.

Proposition 1.75. The tangent bundle TM of any given manifold is, in fact, a vector bundle of rank n.

Proof. We need to explain the topology on TM, find an atlas, and show that we can locally trivialize it as a vector bundle. Fix a countable atlas (cf. Exercise 1.12)

$$\mathcal{A} = \{(\varphi_i = (x_i^1, \dots, x_i^n), U_i) \mid i \in A\}$$

on M. Since π is assumed to be smooth and hence continuous, the pre-images $\{\pi^{-1}(U_i) \mid i \in A\}$ form an open covering of TM. Taking pre-images under π of a basis of the topology on M is not enough to explain the topology on M. For $i \in A$ consider the maps

$$\begin{aligned} &\psi_i : \pi^{-1}(U_i) \to \varphi_i(U_i) \times \mathbb{R}^n, \\ &\qquad \psi_i : v \mapsto (\varphi_i(\pi(v)), v(x_i^1), \dots, v(x_i^n)) = (\varphi_i(\pi(v)), d\varphi_i(v)). \end{aligned} \tag{1.10}$$

and observe that each ψ_i is a bijection. We can think of the above maps as candidates for a local trivialization that, via a chart on the manifold M itself, has its target space changed as in in the following diagram

$$\begin{array}{ccc} \pi^{-1}(U_i) & & \\ & & \\ U_i \times \mathbb{R}^n & & \psi_i \\ \downarrow {\scriptstyle \varphi_i \times \mathrm{id}_{\mathbb{R}^n}} & & \\ \varphi_i(U_i) \times \mathbb{R}^n & \hookleftarrow & \end{array} \tag{1.11}$$

We define a basis of the topology on TM as

$$\{\psi_i^{-1}(V) \mid i \in A,\ V \subset \varphi_i(U_i) \times \mathbb{R}^n \text{ open}\}$$

which is precisely the coarsest topology on TM, such that all maps ψ_i, $i \in A$, are homeomorphisms. Since A is a countable set and the topology in each U_i has a countable basis, it follows that the so-defined topology on TM is countable. To see that it is also Hausdorff, consider for $p \neq q \in TM$ the points $\psi_i(p)$ and $\psi_j(q)$ for fitting $i, j \in A$. If $U_i \cap U_j = \emptyset$, we can separate p and q by $\pi^{-1}(U_i)$ and $\pi^{-1}(U_j)$. For $U_i \cap U_j \neq \emptyset$, observe that each space $\varphi_i(U_i) \times \mathbb{R}^n$ is Hausdorff and we can thus find open neighbourhoods of $\psi_i(p)$ and $\psi_j(q)$ in $\varphi_i(U_i \cap U_j) \times \mathbb{R}^n$ that separate these points. The pre-images under ψ_i of these sets will then separate p and q. Next consider the transition functions (thought of as change of coordinates) of the ψ_i's. For $U_i \cap U_j \neq \emptyset$ we have (recall Example 1.50)

$$\begin{aligned} &\psi_i \circ \psi_j^{-1} : \varphi_j(U_i \cap U_j) \times \mathbb{R}^n \to \varphi_i(U_i \cap U_j) \times \mathbb{R}^n, \\ &\qquad (u, w) \mapsto ((\varphi_i \circ \varphi_j^{-1})(u), d(\varphi_i \circ \varphi_j^{-1})_u(w)). \end{aligned} \tag{1.12}$$

Since the transition functions $\varphi_i \circ \varphi_j^{-1}$ are smooth it follows that the countable set

$$\{(\psi_i, \pi^{-1}(U_i)) \mid i \in A\}$$

[11] Also simply called **tangent space**, without the "at p" part.

$\{(\psi_i, \pi^{-1}(U_i)) \mid i \in A\}$ defines a countable atlas on TM. The vector bundle structure on TM is explained by the local trivializations $(\varphi_i^{-1} \times \mathrm{id}_{\mathbb{R}^n}) \circ \psi_i$, $i \in A$, cf. (1.11). □

Remark 1.76. Compare the proofs of Proposition 1.73 and 1.75 for similarities and differences. Note that, by Proposition 1.73 the structure of a vector bundle in $TM \to M$ is uniquely determined by requiring that the transition functions are given by (1.12). Also note that the transition functions of the local trivializations $(\varphi_i^{-1} \times \mathrm{id}_{\mathbb{R}^n}) \circ \psi_i$ are given by

$$(\varphi_i^{-1} \times \mathrm{id}_{\mathbb{R}^n}) \circ \psi_i \circ \Big((\varphi_j^{-1} \times \mathrm{id}_{\mathbb{R}^n}) \circ \psi_j\Big)^{-1} = (\mathrm{id}_M, d(\varphi_i \circ \varphi_j^{-1})),$$

that is the matrix parts are differentials of the transition functions of the charts on M.

Remark 1.77. Let M be a smooth manifold and $\varphi = (x^1, \dots, x^n)$ a local coordinate system covering $p \in M$. Let $v \in T_pM$,

$$v = \sum_{i=1}^{n} v^i \left.\frac{\partial}{\partial x^i}\right|_p .$$

Observe that with ψ as in (1.10)

$$\psi(v) = (x^1(p), \dots, x^n(p), v^1, \dots, v^n),$$

meaning that the vector part of $\psi(v)$ consists of the prefactors of v in the basis $\left\{ \left.\frac{\partial}{\partial x^i}\right|_p ,\ 1 \le i \le n \right\}$.

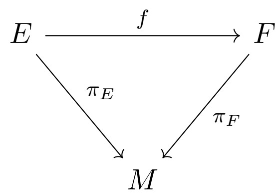
Exercise 1.78. Consider S^n with atlas the stereographic projections as in Example 1.18 (i). Explicitly calculate the corresponding transition functions (1.12) in the tangent bundle TS^n.

Given a smooth manifold, one might ask how "bad" the tangent bundle might look like. For this question we first need to clarify when two vector bundles are considered isomorphic.

Definition 1.79. Let $\pi_E : E \to M$ and $\pi_F : F \to M$ be vector bundles over smooth manifolds M. Then a **smooth vector bundle homomorphism**[12] is a smooth map between the total spaces

$$f : E \to F,$$

such that the diagram



commutes and f is fibrewise linear. The last condition means that for each $p \in M$,

$$f|_{E_p} : E_p \to F_p$$

is a linear map.

Definition 1.80. Two vector bundles $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M$ are **isomorphic** if there exists a diffeomorphism $F : E_1 \to E_2$ that is a smooth vector bundle map, so that

$$F|_{E_{1p}} : E_{1p} \to E_{2p}$$

is a linear isomorphism for all $p \in M$.

[12] a.k.a. "smooth vector bundle map"

Definition 1.81. A vector bundle of rank k, $E \to M$, is called **trivializable** if it is isomorphic to $M \times \mathbb{R}^k \to M$ equipped with the canonical projection onto M.

Lemma 1.82. Assume that $E \to M$ is trivializable. Then there exists a nowhere vanishing section $s \in \Gamma(E)$.

Proof. Exercise. □

The best case scenario we can expect for the tangent bundle of a smooth manifold is that it is trivializable. This is in general false. An example of a smooth manifold that with non-trivializable tangent bundle is S^2. It follows from the "hairy ball theorem"[13] [M]. But there are non-trivial examples:

Exercise 1.83. Show that TS^1 is trivializable and, hence, as a smooth manifold isomorphic to the cylinder $S^1 \times \mathbb{R}$. Draw a sketch of the isomorphism.

We have now all tools at hand to define vector fields on smooth manifolds.

Definition 1.84. Sections in the tangent bundle of a smooth manifold, $\Gamma(TM)$, are called **vector fields**. For $X \in \Gamma(TM)$ we will denote the value of X at $p \in M$ by X_p. For $U \subset M$ open, we will call elements of $\Gamma(TM|_U)$ **local vector fields**, or simply vector fields if the setting does not explicitly use the locality property. We will use the notations

$$\mathfrak{X}(M) := \Gamma(TM)$$

and

$$\mathfrak{X}(U) := \Gamma(TM|_U)$$

for $U \subset M$ open.

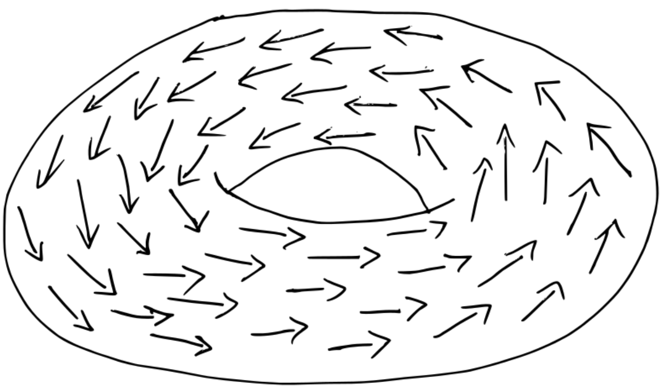


Figure 19: A vector field on the 2-torus.

Remark 1.85. For a smooth manifold M and $U \subset M$ open, the two vector spaces T_pU and T_pM are canonically isomorphic via restriction of charts for all $p \in U$. In the following we will omit using T_pU and instead write T_pM, e.g. if we want to denote the action of a tangent vector on a function $f \in C^\infty(U)$, $v(f)$, we will write $v \in T_pM$ and not $v \in T_pU$.

[13]German: "Satz vom Igel"

Remark 1.86. Vector fields, similar to tangent vectors, act on $C^\infty(M)$ by

$$X(f)(p) := X_p(f) = df(X_p).$$

Thus we may write $X(f) = df(X) \in C^\infty(M)$. On the other hand, a map of the form $X : M \to TM$, $p \mapsto X_p \in T_pM$, is a vector field if $X(f) : p \mapsto df_p(X_p)$ is smooth for all $f \in C^\infty(M)$.

Recall that in Proposition 1.46 we have shown that in local coordinates $(x^1, \ldots, x^n)$ the tangent vectors $\frac{\partial}{\partial x^i}\Big|_p$, $1 \le i \le n$, form a basis of T_pM. We want to have a similar result for the local form of vector fields in $\Gamma(TM|_U)$ for U the chart neighbourhood of the local coordinates x^i.

Definition 1.87. Let $(\varphi = (x^1, \ldots, x^n), U)$ be a chart on a smooth manifold M. The corresponding **coordinate vector fields** are defined as

$$\frac{\partial}{\partial x^i} \in \mathfrak{X}(U), \quad \frac{\partial}{\partial x^i} : p \mapsto \frac{\partial}{\partial x^i}\Big|_p.$$

Proposition 1.88. Let $(\varphi = (x^1, \ldots, x^n), U)$ be a chart on a smooth manifold M and $X \in \mathfrak{X}(U)$. With $X^i := X(x_i) \in C^\infty(U)$ we have

$$X = \sum_{i=1}^n X^i \frac{\partial}{\partial x^i}.$$

On the other hand for any choice of smooth functions $f^i \in C^\infty(U)$, $1 \le i \le n$,

$$\sum_{i=1}^n f^i \frac{\partial}{\partial x^i} \in \mathfrak{X}(U).$$

Proof. The first claim follows from the fact that for any $p \in U$ fixed, $X_p = \sum_{i=1}^n X_p(x^i) \frac{\partial}{\partial x^i}\Big|_p$, which follows from Proposition 1.46. The second claim follows from the fact that each $\frac{\partial}{\partial x^i}$ is a vector field on U and Exercise 1.70 (i). ☐

Exercise 1.89.

(i) Prove the statements in Remark 1.86.

(ii) Construct a vector field $X \in \mathfrak{X}(S^2)$ with precisely one bald spot, meaning that there should exists precisely one $p \in S^2$, such that $X_p = 0$.

There is an alternative, but equivalent, way of introducing vector fields on smooth manifolds, see [O]. Recall that in Definition 1.34 we have initially defined tangent vectors to be linear maps from $C^\infty(M)$ to $\mathbb{R}$ satisfying a Leibniz rule. Vector fields can be introduced similarly using the concept of derivations from differential algebra.

Definition 1.90. Let A be an algebra over a field K. A **derivation** of a A is a K-linear map $D : A \to A$ that fulfils the Leibniz rule

$$D(ab) = D(a)b + aD(b)$$

for all $a, b \in A$. The set of all derivations of A is denoted by $\mathrm{Der}(A)$. If A is commutative, $\mathrm{Der}(A)$ is an A module.

Recall that the smooth functions on a manifold, $C^\infty(M)$, is an $\mathbb{R}$-algebra.

Proposition 1.91. Let M be a smooth manifold. Then vector fields on M are precisely the derivations of $C^\infty(M)$, meaning that $\mathfrak{X}(M)$ and $\mathrm{Der}(C^\infty(M))$ are isomorphic as $C^\infty(M)$ modules.

Proof. The map

$$\iota : \mathfrak{X}(M) \to \mathrm{Der}(C^\infty(M)), \quad X \mapsto (f \mapsto X(f)),$$

is a $C^\infty(M)$ module map. Injectivity of ι follows from $X = 0$ if and only if $X(f) = 0$ for all $f \in C^\infty(M)$ (see proof of Proposition 1.46 if you have problems seeing that fact). For surjectivity we define for a given derivation D a vector field X^D via

$$D \mapsto X^D, \quad X^D_p(f) = D(f)(p)$$

for all $p \in M$ and all $f \in C^\infty(M)$. By Remark 1.86 we know that X^D is in fact a smooth vector field. The map $D \mapsto X^D$ is precisely the inverse of ι. □

We now know the algebraic properties of vector fields as derivations and we know how to write down and calculate with vector fields locally. The following lemma describes explicitly how vector fields behave under a change of coordinates.

Lemma 1.92. Let M be a smooth manifold and let $(\varphi = (x^1, \ldots, x^n), U)$, $(\psi = (y^1, \ldots, y^n), V)$ be charts on M such that $U \cap V \neq \emptyset$. For $X \in \mathfrak{X}(M)$ fixed, we have on $U \cap V$ the following forms of X in local coordinates

$$X = \sum_{i=1}^n X(x^i) \frac{\partial}{\partial x^i}$$

and

$$X = \sum_{i=1}^n X(y^i) \frac{\partial}{\partial y^i}.$$

If we understand $d(\psi \circ \varphi^{-1}) : \varphi(U \cap V) \to \mathrm{GL}(n)$ as a matrix-valued function which associates each point $u \in \varphi(U \cap V)$ the Jacobi matrix of $\psi \circ \varphi^{-1}$ at u we obtain

$$d(\psi \circ \varphi^{-1})_u \cdot \begin{pmatrix} X(x^1) \\ \vdots \\ X(x^n) \end{pmatrix}\Bigg|_{\varphi^{-1}(u)} = \begin{pmatrix} X(y^1) \\ \vdots \\ X(y^n) \end{pmatrix}\Bigg|_{\psi^{-1}(u)}$$

for all $u \in U \cap V$.

Proof. Follows from the definition of the Jacobi matrix and the coordinate vector fields. □

If the above formula looks difficult to you, calculate some examples for $M = \mathbb{R}^n$, $\varphi = \mathrm{id}_{\mathbb{R}^n}$, and $\psi : \mathbb{R}^n \to \mathbb{R}^n$ any diffeomorphism. Having defined vector fields and coordinate vector fields, we can now properly define differentials of smooth maps, compared to our pointwise Definition 1.49.

Definition 1.93. Let M, N be smooth manifolds and $F : M \to N$ be a smooth map. The **differential of** F is defined as the smooth map

$$dF : TM \to TN, \quad dF|_{\pi^{-1}(p)} = dF_p \quad \forall p \in M.$$

The above equation just means that pointwise, dF is given by its differential as in Definition 1.49. Thus, in local coordinates $(x^1, \ldots, x^m)$ of M and $(y^1, \ldots, y^n)$ of N with appropriate domain we have

$$dF\left(\frac{\partial}{\partial x^i}\right) = \sum_{j=1}^n \frac{\partial F^j}{\partial x^i} \frac{\partial}{\partial y^j}, \quad F^j = y^j \circ F, \quad \forall 1 \leq i \leq m.$$

The (non-pointwise) Jacobi matrix in given local coordinates is defined similarly by allowing the basepoint to vary and, as a map from chart neighbourhoods in M to $\mathrm{GL}(n)$, is also smooth.

Exercise 1.94. Check using local coordinates on TM and TN that dF and the Jacobi matrix as in Definition 1.93 are actually smooth.

On the vector fields on a smooth manifold we have the structure of a **Lie**[14] **algebra**. Before describing this concept in detail, consider for two derivations $X, Y \in \mathrm{Der}(A)$ of an algebra A the commutator of X and Y

$$[X, Y] := XY - YX.$$

Exercise 1.95. Show that $[X, Y] \in \mathrm{Der}(A)$.

By Proposition 1.91 there must be an analogue construction on the set of vector fields on a smooth manifold.

Definition 1.96. Let V be a real vector space. A **Lie bracket** on V is a skew-symmetric bilinear map

$$[\cdot, \cdot] : V \times V \to V, \quad (X, Y) \mapsto [X, Y]$$

that fulfils the[15] **Jacobi identity**

$$[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]]$$

for all $X, Y, Z \in V$. A vector space V together with a Lie bracket is called **Lie algebra**.

Exercise 1.97.

(i) $[\cdot, \cdot]$ on $\mathrm{Der}(A)$ is a Lie bracket.

(ii) Show that the Jacobi identity as defined in Definition 1.96 is equivalent to

$$\sum_{\text{cyclic}} [X, [Y, Z]] = 0,$$

for all $X, Y, Z \in V$, where $\sum\limits_{\text{cyclic}}$ stands for the cyclic sum.

Proposition 1.98. The bilinear map on vector fields on a smooth manifold M

$$\begin{aligned} &[\cdot, \cdot] : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M), \quad (X, Y) \mapsto [X, Y], \\ &[X, Y](f) := X(Y(f)) - Y(X(f)) \quad \forall X, Y \in \mathfrak{X}(M)\ \forall f \in C^\infty(M), \end{aligned}$$

is a Lie bracket on the vector space $\mathfrak{X}(M)$.

Proof. Follows from Exercise 1.97 (i). □

Note that $[X, Y]_p(f) = X_p(Y(f)) - Y_p(X(f))$ for all $p \in M$, $f \in C^\infty(M)$, $X, Y \in \mathfrak{X}(M)$. From real analysis we know that partial derivatives commute. We can formulate a similar result for smooth manifolds with the help of the Lie algebra structure on $\mathfrak{X}(M)$.

Lemma 1.99. Let M be a smooth manifold and $(x^1, \ldots, x^n)$ be local coordinates. Then

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0$$

for all $1 \leq i \leq n$, $1 \leq j \leq n$.

[14] Sophus Lie (1842 – 1899)

[15] Careful: There is more than one Jacobi identity, e.g. graded Jacobi identities.

Proof. Exercise. □

Exercise 1.100. Show that $[X, fY] = df(X)Y + f[X, Y]$ for all $X, Y \in \mathfrak{X}(M)$, $f \in C^\infty(M)$.

Definition 1.101. Let $\phi : M \to N$ be a smooth map and let $X \in \mathfrak{X}(M)$. The smooth map

$$M \ni p \mapsto (d\phi(X))_p = d\phi_p(X_p) \in T_{\phi(p)}N$$

is called a **vector field along** ϕ.

Note that $d\phi X$ sends smooth functions on N to smooth functions on M via

$$(d\phi X)(f) = X(f \circ \phi) \in C^\infty(M)$$

for all $f \in C^\infty(N)$.

Definition 1.102. Let $I \subset \mathbb{R}$ be an interval (equipped with canonical coordinate t), M a smooth manifold, and $\gamma : I \to M$ a smooth curve. The **velocity vector field** (or simply **velocity**) of γ is the vector field along γ

$$\gamma' := d\gamma\left(\frac{\partial}{\partial t}\right), \quad t \mapsto \gamma'(t).$$

Note that the explicit form of $\gamma'(t)$ depends on the local coordinates $\varphi = (x^1, \ldots, x^n)$ on M:

$$\gamma'(t) = \sum_{i=1}^{n} \frac{\partial \gamma^i}{\partial t}(t) \left.\frac{\partial}{\partial x^i}\right|_{\gamma(t)} \in T_{\gamma(t)}M$$

for all $t \in I$, where $\gamma^i = x^i(\gamma)$ for all $1 \leq i \leq n$

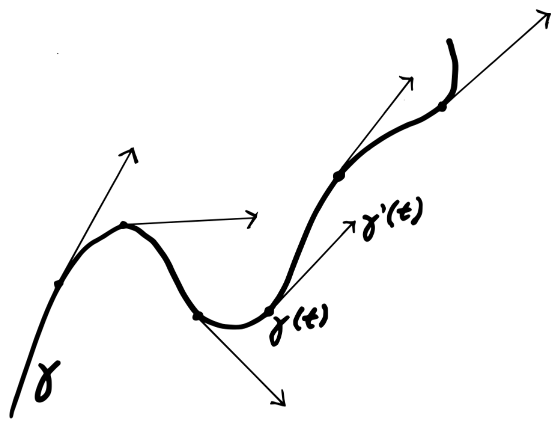


Figure 20: The velocity vector field of a curve γ.

Now that we have defined the velocity of smooth curves in smooth manifolds, we can relate our initial definition of tangent vectors in $\mathbb{R}^n$ in Remark 1.33 to tangent vectors for general smooth manifolds as follows.

Lemma 1.103. Let M be a smooth manifold, $v \in T_pM$, and $f \in C^\infty(M)$. Then

$$v(f) = \frac{\partial (f \circ \gamma)}{\partial t}(0)$$

for every smooth curve $\gamma : I \to M$, $0 \in I$, with $\gamma(0) = p$ and $\gamma'(0) = v$.

Proof. Follows from the chain rule for differentials of smooth maps. $\square$

Recall the definition of an integral curve of a vector field on an open set of $\mathbb{R}^n$. There is, of course, a similar concept for smooth manifolds in general.

Definition 1.104. Let $X \in \mathfrak{X}(M)$ for a smooth manifold M. An **integral curve** of X at $p \in M$ is a smooth curve $\gamma : I \to M$, where $I \subset \mathbb{R}$ is an interval, $0 \in I$, such that $\gamma(0) = p$ and

$$\gamma'(t) = X_{\gamma(t)}$$

for all $t \in I$. An integral curve $\gamma : I \to M$ of X is called **maximal** if there is no interval $\widetilde{I} \supset I$, such that $\widetilde{I} \setminus I \neq \emptyset$ and there exists an integral curve $\widetilde{\gamma} : \widetilde{I} \to M$ of X with $\widetilde{\gamma}|_I = \gamma$. A vector field X is called **complete** if every maximal integral curve $\gamma : I \to M$ is defined on $I = \mathbb{R}$.

If we omit the term "at p" for integral curves, we also mean that the interval I does not necessarily contain 0.

Example 1.105. Consider $X \in \mathfrak{X}(\mathbb{R}^2)$ that is in canonical coordinates $(u^1, u^2) = (x, y)$ given by

$$X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

Its integral curves at any point $(x_0, y_0) \in \mathbb{R}^2$ are of the form

$$\gamma : t \mapsto \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

Exercise 1.106.

(i) Write down the vector field X and its integral curves in Example 1.105 in polar coordinates. Is X complete?

(ii) Construct a vector field on $\mathbb{R}^2 \setminus \{0\}$ with precisely one periodic maximal integral curve that is not a constant curve and no other periodic maximal integral curves.

Remark 1.107. While we have defined integral curves of vector fields, at this point we do not know if they always exists and whether they are unique or not. With the help of the theory of ordinary differential equations we obtain such results. Firstly note that locally, i.e. in any given local coordinates, the equation

$$\gamma' = X_\gamma$$

is an ordinary, in general non-linear, differential equation. Thus, for any given vector field $X \in \mathfrak{X}(M)$ and any $p \in M$ there exists an integral curve $\gamma : I \to M$ of X at p. If $\gamma : I \to M$ and $\widetilde{\gamma} : \widetilde{I} \to M$ are two integral curves of X at p, they coincide on $I \cap \widetilde{I}$ which, by definition, is never empty. For each $p \in M$, there exists a unique maximal integral curve of X at p. Furthermore, for a the integral curves of X at p depend locally smoothly on $p \in M$. The proofs of these results need some care when an integral curves leaves a given coordinate neighbourhood but are otherwise identical to the case $M \subset \mathbb{R}^n$ open. For literature on the subject of ordinary differential equations and dynamical systems see e.g. [A1, A2]

In general it is a very difficult question whether a given vector field in $\mathfrak{X}(M)$ is complete, at least if M is not compact. For compact smooth manifolds M we have the following result.

Proposition 1.108. Vector fields on compact smooth manifolds are complete.

Proof. [A1] Chapter 2.6 together with the fact that since M is compact, one can for any given atlas $\mathcal{A}$ on M assume without loss of generality that $\mathcal{A}$ is finite, and we have that the closure of the chart neighbourhoods of $\mathcal{A}$ are compact in M. □

For M not compact we still have the following result on vector fields with compact support.

Proposition 1.109. Let $X \in \mathfrak{X}(M)$ be a vector field with compact support, meaning that

$$\operatorname{supp}(X) = \overline{\{p \in M \mid X_p \neq 0\}} \subset M$$

is compact. Then X is complete.

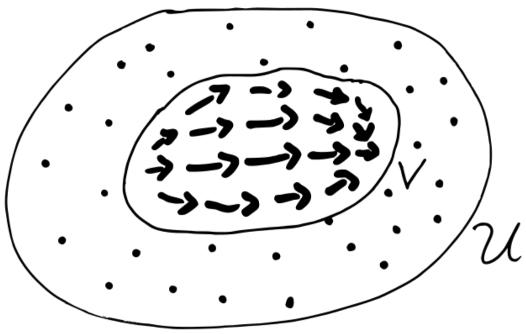


Figure 21: A sketch of a vector field with compact support $V \subset U$.

Proof. Exercise. [Hint: Try proving this for $M = \mathbb{R}^n$ first.] □

Definition 1.110. A **local one parameter group of diffeomorphisms** on a smooth manifold M is a smooth map

$$\varphi : I \times U \to M, \quad (t, p) \mapsto \varphi_t(p),$$

such that $I \subset \mathbb{R}$ is an interval containing $0 \in \mathbb{R}$, $U \subset M$ is open, $\varphi_0 = \mathrm{id}_U$, $\varphi_t : M \to M$ is a diffeomorphism for all $t \in I$, and

$$\varphi_{s+t}(p) = \varphi_s(\varphi_t(p))$$

for all $p \in U$ and all $s, t \in I$ with $(s + t) \in I$ and $\varphi_t(p) \in U$. A **one parameter group of diffeomorphisms** is a local one parameter group of diffeomorphisms with $I = \mathbb{R}$ and $U = M$.

Local one parameter groups of diffeomorphisms on smooth manifolds are closely related to vector fields and their integral curves. For any given vector field on a smooth manifold we can attempt to consider all integral curves of X "at once". This leads to the following concept.

Definition 1.111. A **local flow** of a vector field $X \in \mathfrak{X}(M)$ is a smooth map

$$\varphi : I \times U \to M, \quad (t, p) \mapsto \varphi_t(p),$$

for some interval $I \subset \mathbb{R}$ containing $0 \in \mathbb{R}$ and an open set $U \subset M$, such that $\varphi_0 = \mathrm{id}_U$ and for every $p \in U$ fixed, the smooth curve

$$t \mapsto \varphi_t(p)$$

is an integral curve of X. This just means that

$$\frac{\partial}{\partial t}(\varphi_t(p)) = X_{\varphi_t(p)}.$$

We say that a local flow of X is defined near a point $p \in M$ if $p \in U$. A local flow of X is called **(global) flow** of X if $I = \mathbb{R}$ and $U = M$.

By saying that a local flow $\varphi : I \times U \to M$ is near some point $p \in M$ we mean that $p \in U$.

Lemma 1.112. Every vector field on M admits a local flow near any given point $p \in M$.

Proof. Let $X \in \mathfrak{X}(M)$ and $p \in M$ arbitrary but fixed. Choose a bump function $b : M \to \mathbb{R}$ such that on some open neighbourhood $U \subset M$ of p, $b|_U \equiv 1$. The maximal integral curves at p of bX are each defined on $\mathbb{R}$ by Proposition 1.109 and depend smoothly on $p \in M$ by Remark 1.107. This already shows that vector fields with compact support admit a global flow φ. Fix $\varepsilon > 0$ and choose an open subset $V \subset U$, such that V is an open neighbourhood of p and for all $q \in V$ and all $t \in (-\varepsilon, \varepsilon)$ $\varphi_t(q) \in U$. Geometrically this mean that the set V is not moved out of U by the flow of bX for $|t| < \varepsilon$. Since X and bX coincide on U, their integral curves at all $q \in V$ for $I = (-\varepsilon, \varepsilon)$ also coincide. Hence, the flow φ of bX restricted to $(-\varepsilon, \varepsilon) \times V$ is a local flow of X. □

We already see that Definition 1.111 and 1.110 look similar. They are connected as follows.

Proposition 1.113. Local flows of vector fields are local one parameter groups of diffeomorphisms.

Proof. It suffices to show that for a given vector field X with two integral curves $\gamma : (a, b) \to M$ at $p = \gamma(0)$ and $\widetilde{\gamma} : (\widetilde{a}, \widetilde{b}) \to M$ with $\gamma(s) = \widetilde{\gamma}(0)$ for some $s \in (a, b)$ we have

$$\gamma(s + t) = \widetilde{\gamma}(t)$$

for all t, such that $(s + t) \in (a, b)$ and $t \in (\widetilde{a}, \widetilde{b})$. This means that $\widetilde{\gamma}$ extends γ and follows from the fact that $t \mapsto \gamma(s + t)$ is an integral curve of X (for s small enough) and uniqueness of local solutions:

$$(\gamma(s + \cdot))'(t) \overset{\text{chain rule}}{=} \gamma'(s + t) = X_{\gamma(s+t)} = X_{(\gamma(s+\cdot))(t)}.$$

Hence, for a local flow ϕ of X near p we obtain

$$\phi_t(\phi_s(p)) = \phi_t(\gamma(s)) = \widetilde{\gamma}(t) = \gamma(s + t) = \phi_{s+t}(p).$$

□

The following is an immediate consequence of Proposition 1.113.

Corollary 1.114. Assume that $X \in \mathfrak{X}(M)$ is complete. Then its flow is a one parameter group of diffeomorphisms.

In fact, one can prove that for any vector field $X \in \mathfrak{X}(M)$ the set

$$\bigcup_{p \in M} (I_p \times \{p\}) \subset (\mathbb{R} \times M)$$

is open, where I_p is the uniquely determined interval for the maximal integral curve $\gamma : I_p \to M$ of X starting at $\gamma(0) = p \in M$. For X complete, $I_p = \mathbb{R}$ for all $p \in M$ and, hence, the maximal domain of definition of any local flow of X is $\mathbb{R} \times M$, meaning X has a global flow. For the proof see [G, S. 1.10.9].

Example 1.115. Translations in $\mathbb{R}^n$ are of the form $A_v : (p, v) \mapsto p + v$ where $v = (v^1, \ldots, v^n)$ is the translation vector. Consider the constant vector field

$$\sum_{i=1}^{n} v^i \frac{\partial}{\partial u^i}$$

with global flow

$$\varphi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n, \quad (t, p) \mapsto p + tv.$$

We see that $\varphi(1, p) = A_v(p)$ for all $p \in \mathbb{R}^n$.

We have seen that if the local flow of a vector field is a local one parameter group of diffeomorphisms. The other direction is true as well.

Exercise 1.116. Let $A \in \mathrm{SO}(2)$ be fixed. Find $X \in \mathfrak{X}(\mathbb{R}^2)$, such that its global flow φ fulfils $\varphi_1(p) = Ap$ for all $p \in \mathbb{R}^2$. Can this always be achieved for any $A \in \mathrm{O}(2)$?

We have seen that local flows of vector fields are local one parameter groups of diffeomorphisms. The converse statement is also true.

Definition 1.117. Let $\varphi : I \times U \to M$ be a local one parameter group of diffeomorphisms. The **infinitesimal generator of** φ is defined to be the map

$$U \ni p \mapsto \left.\frac{\partial}{\partial t}\right|_{t=0} (\varphi_t(p)) \in T_pM.$$

[Note: We have secretly used Exercise 1.142, make sure you understand how and why.]

Lemma 1.118. Infinitesimal generators of local one parameter group of diffeomorphisms $\varphi : I \times U \to M$ are local vector fields in $\mathfrak{X}(U)$. Infinitesimal generators of one parameter groups of diffeomorphisms $\varphi : \mathbb{R} \times M \to M$ are complete.

Proof. Since any local one parameter group of diffeomorphisms is smooth, the map $X : p \mapsto X_p := \left.\frac{\partial}{\partial t}\right|_{t=0} (\varphi_t(p))$ is smooth, i.e. $X \in \mathfrak{X}(U)$. Hence, for any one parameter group of diffeomorphisms $\varphi : \mathbb{R} \times M \to M$, X is a vector field in $\mathfrak{X}(M)$. Its integral curves at $p \in M$ are given by

$$t \mapsto \varphi_t(p)$$

and are defined for all $t \in \mathbb{R}$. This means that X is complete. □

Recall the definition of the Lie bracket on $\mathfrak{X}(M)$, cf. Proposition 1.98. We know the algebraic motivation for it by considering vector fields as derivations of $C^\infty(M)$. But what does $[X, Y]$ for $X, Y \in \mathfrak{X}(M)$ stand for geometrically? To answer this question we must define the pushforward and pullback of vector fields under diffeomorphisms.

Definition 1.119. Let $F : M \to N$ be a diffeomorphism and let $X \in \mathfrak{X}(M)$, $Y \in \mathfrak{X}(N)$. The **pushforward** of X under F is the vector field $F_*X \in \mathfrak{X}(N)$ given by

$$(F_*X)_q := dF_{F^{-1}(q)} \left(X_{F^{-1}(q)} \right) \quad \forall q \in N.$$

The **pullback** of Y under F is the vector field $F^*Y \in \mathfrak{X}(M)$ given by

$$(F^*Y)_p := d(F^{-1})_{F(p)} \left(Y_{F(p)} \right) \quad \forall p \in M.$$

Note that $d(F^{-1})_{F(p)} = (dF_p)^{-1}$ for all $p \in M$.

Exercise 1.120. Verify that if $F : M \to N$ is a diffeomorphism and γ is an integral curve of $X \in \mathfrak{X}(M)$, then $F \circ \gamma$ is an integral curve of F_*X. Formulate a version of this statement for local diffeomorphisms.

In order to explain the Lie bracket of vector fields geometrically, we need one more result about the local form of vector fields. Assume that $X \in \mathfrak{X}(M)$ does not vanish everywhere. Then near any point where X does not vanish we can find local coordinates on M in which X has a particularly simple form.

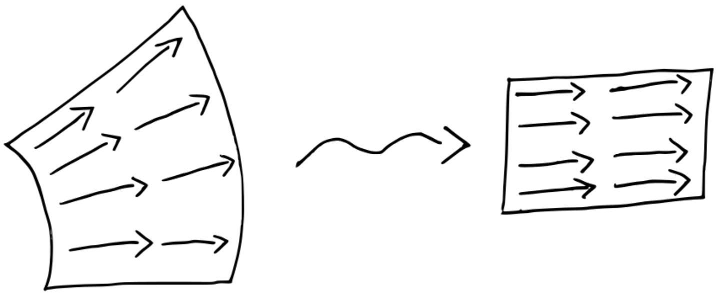


Figure 22: Locally rectifying a vector field.

Proposition 1.121. Let $X \in \mathfrak{X}(M)$ and $p \in M$, such that $X_p \neq 0$. Then there exist local coordinates on an open neighbourhood $U \subset M$ of p, such that X is of the form

$$X_q = \frac{\partial}{\partial x^1}\bigg|_q$$

for all $q \in U$.

Proof. Since X is as a section in TM it is in particular a continuous map and, hence, we can find an some open neighbourhood U of $p \in M$, such that $X_q \neq 0$ for all $q \in U$. Assume without loss of generality that U is contained in a chart neighbourhood. Choose any local coordinate system $\phi = (y^1, \ldots, y^n)$ on U and let $(u^1, \ldots, u^n)$ denote the canonical coordinates on $\mathbb{R}^n$. We can assume without loss of generality, after possibly shrinking U and re-ordering the y^i's, that $\phi_*(X) \in \mathfrak{X}(\phi(U))$ is transversal along the inclusion map $\{u^1 = 0\} \cap \phi(U) \hookrightarrow \mathbb{R}^n$ to $N := \{u^1 = 0\} \cap \phi(U)$, meaning that

$$(\phi_* X)_q \notin T_q N \cong T_q\{u^1 = 0\} \subset T_q\mathbb{R}^n$$

for all $q \in N$. Let, after again possibly shrinking U, $\Phi : I \times \phi(U) \to \mathbb{R}^n$ denote a local flow of $\phi_* X$. Since

$$(\phi_* X)_q = \frac{\partial}{\partial t}\bigg|_{t=0} \Phi_t(q) \neq 0$$

by the transversality condition, we obtain with Theorem 1.54 after possibly shrinking I that

$$F := \Phi|_{I\times N} : I \times N \to \Phi(I \times N)$$

is a diffeomorphism, where we understand I as the "time" part so that $\Phi(t, q) := \Phi_t(q)$, and $\Phi(I \times N) \subset \mathbb{R}^n$ is open. Denoting the canonical coordinates in I by u^1 and in N by $(u^2, \ldots, u^n)$ (which is compatible with the canonical inclusion $I \times N \subset \mathbb{R}^n$), we in particular have

$$dF_{(u^1,u^2,\ldots u^n)}\left(\frac{\partial}{\partial u^1}\bigg|_{(u^1,u^2,\ldots,u^n)}\right) = (\phi_* X)_{\Phi(u^1,u^2,\ldots,u^n)}$$

for all $(u^1, \ldots, u^n) \in I \times N$. Now we can define coordinates on $\phi^{-1}(\phi(U) \cap \Phi(I \times N)) \subset M$ by

$$\psi = (x^1, \ldots, x^n) := F^{-1} \circ \phi : \phi^{-1}(\phi(U) \cap \Phi(I \times N)) \to F^{-1}(\phi(U) \cap (I \times N)) \subset \mathbb{R}^n$$

and obtain for the local formula of X in the local coordinate system ψ and all $q \in \phi^{-1}(\phi(U) \cap \Phi(I \times N))$

$$X_q = \frac{\partial}{\partial x^1}\bigg|_q .$$

□

In local coordinates as the ones constructed in Proposition 1.121, local flows look particularly simple.

Corollary 1.122. Any local flow of X near p as in Proposition 1.121 is, if $X_p \neq 0$, in the local coordinate system $\psi = (x^1, \ldots, x^n)$ of the form

$$\psi(\varphi_t(q)) = \psi(q) + te_1,$$

for all $q \in U$, where e_1 denotes the first unit vector in $\mathbb{R}^n$ in canonical coordinates, for $|t|$ small enough. Furthermore

$$d\varphi_t \left(\left. \frac{\partial}{\partial x^i} \right|_q \right) = \left. \frac{\partial}{\partial x^i} \right|_{\psi^{-1}(\psi(q)+te_1)}$$

for all $q \in U$ and t small enough, where we understand the differential of φ_t for t fixed.

Next we will describe how the Lie algebra structure on vector fields is connected to their local flows. To do so we will need to introduce the following concept.

Definition 1.123. Let M and N be smooth manifolds $\phi : M \to N$ be a smooth map. Two vector fields $X \in \mathfrak{X}(M)$ and $\overline{X} \in \mathfrak{X}(N)$ are called **ϕ-related** if $d\phi(X) = \overline{X}_\phi$. One then writes $X \sim_\phi \overline{X}$. Equivalently, $X \sim_\phi \overline{X}$ if $X(f \circ \phi) = Y(f) \circ \phi$ for all $f \in C^\infty(N)$.

We see that for $\phi : M \to N$ an embedding and any $X \in \mathfrak{X}(M)$, $d\phi(X)$, viewed as vector field along ϕ, can be locally extended to a smooth vector field $\overline{X} \in \mathfrak{X}(N)$, such that $d\phi(X) = \overline{X}_\phi$. This means that, locally, we can find a ϕ-related vector field to X. For the next lemma, the motivation is the case where ϕ is a change of coordinates on a smooth manifold.

Lemma 1.124. Let $\phi : M \to N$ be a smooth map, $X, Y \in \mathfrak{X}(M)$ and $\overline{X}, \overline{Y} \in \mathfrak{X}(N)$, such that $X \sim_\phi \overline{X}$ and $Y \sim_\phi \overline{Y}$. Then $[X, Y] \sim_\phi [\overline{X}, \overline{Y}]$.

Proof. Let $f \in C^\infty(N)$ arbitrary. Then

$$\begin{aligned}
[X, Y](f \circ \phi) &= X(Y(f \circ \phi)) - Y(X(f \circ \phi)) \\
&= X(\overline{Y}(f) \circ \phi) - Y(\overline{X}(f) \circ \phi) \\
&= (\overline{X}(\overline{Y}(f)) - \overline{Y}(\overline{X}(f))) \circ \phi.
\end{aligned}$$

□

Lemma 1.124 means for ϕ a change of coordinates that the Lie algebra structure on vector fields is compatible with changing coordinates in the sense that their Lie brackets are also related by the same change of coordinates. Globally we have the following.

Corollary 1.125. For diffeomorphisms $F : M \to N$,

$$F_*[X, Y] = [F_*X, F_*Y]$$

for all $X, Y \in \mathfrak{X}(M)$ and

$$F^*[\overline{X}, \overline{Y}] = [F^*\overline{X}, F^*\overline{Y}]$$

for all $\overline{X}, \overline{Y} \in \mathfrak{X}(N)$.

Proof. Follows from Definition 1.119 and Lemma 1.124. □

Remark 1.126. In the case that ϕ is an embedding and $\dim(M) < \dim(N)$, Lemma 1.124 also implies that (locally and globally) $[\overline{X}, \overline{Y}] \circ \phi$ does **not** depend on the (local) extensions of $\overline{X} \circ \phi$ and $\overline{Y} \circ \phi$ to vector fields on in N open neighbourhoods of points in $\phi(M)$.

Now we have cleared up all technical difficulties and can proof the following statement.

Means: Vector fields tangent to a submanifold have Lie bracket tangent to submanifold.

Proposition 1.127. Let $X, Y \in \mathfrak{X}(M)$ and for $p \in M$ arbitrary but fixed let $\varphi : I \times U \to M$ be a local flow of X near p. Then

$$[X, Y]_p = \frac{\partial}{\partial t}\bigg|_{t=0} (\varphi_t^* Y)_p.$$

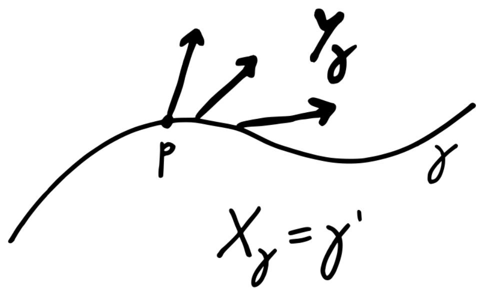


Figure 23: A sketch of Y along an integral curve γ through p of X.

Proof. First observe that the right hand side of the above formula is actually a well-defined expression. This follows from $(\varphi_t^* Y)_p \in T_pM$ for all $t \in I$ and the fact that T_pM is a real vector space. In the following, $d\varphi_t$ is to be understand as differential of φ_t for $t \in I$ fixed. First assume that $X_p \neq 0$. Without loss of generality we can, with the help of Proposition 1.121, assume that we have chosen local coordinates $(x^1, \dots, x^n)$ on $U \subset M$ with $p \in M$, such that $X_q = \frac{\partial}{\partial x^1}\big|_q$ for all $q \in U$. Recall that, since φ is a local one parameter group of diffeomorphisms, $\varphi_{-t} = \varphi_t^{-1}$ whenever defined. Hence for $|t|$ small enough we have

$$(\varphi_t^* Y)_p = d(\varphi_t^{-1})_{\varphi_t(p)} \left(Y_{\varphi_t(p)} \right) = (d\varphi_{-t})_{\varphi_t(p)} \left(Y_{\varphi_t(p)} \right).$$

Observe that

$$(d\varphi_{-t})_{\varphi_t(p)} : \frac{\partial}{\partial x^i}\bigg|_{\varphi_t(p)} \mapsto \frac{\partial}{\partial x^i}\bigg|_{\psi^{-1}(\psi(\varphi_t(p)) - te_1)} = \frac{\partial}{\partial x^i}\bigg|_{\psi^{-1}(\psi(p) + te_1 - te_1)} = \frac{\partial}{\partial x^i}\bigg|_p.$$

In the local coordinates $(x^1, \dots, x^n)$, Y is of the form

$$Y_q = \sum_{i=1}^n Y^i(q) \frac{\partial}{\partial x^i}\bigg|_q$$

for all $q \in U$. Thus

$$(\varphi_t^* Y)_p = \sum_{i=1}^n Y^i(\varphi_t(p)) \frac{\partial}{\partial x^i}\bigg|_p$$

and, hence,

$$\frac{\partial}{\partial t}\bigg|_{t=0} (\varphi_t^* Y)_p = \sum_{i=1}^n dY^i \left(\frac{\partial}{\partial x^1}\bigg|_p \right) \frac{\partial}{\partial x^i}\bigg|_p$$

which coincides with $[X, Y]_p = \left[\frac{\partial}{\partial x^1}, Y\right]_p$ by Lemma 1.99 and Exercise 1.100.

Next assume that $X_p = 0$. If $X_q = 0$ for all q in an open neighbourhood U of p, the local flow of X restricted to U will be the identity for all $t \in I$. Hence,

$$\left.\frac{\partial}{\partial t}\right|_{t=0} (\varphi_t^* Y)_p = 0.$$

For any $f \in C^\infty(M)$ observe that $X(f)$ vanishes on U and thus

$$[X, Y]_p(f) = X_p(Y(f)) - Y_p(X(f)) = 0.$$

Lastly assume that $X_p = 0$ and X does not vanish identically on some open neighbourhood of p. Let $U \subset M$ be a compactly embedded open neighbourhood of p and choose a sequence $\{p_n\}_{n \in \mathbb{N}}$, $\lim_{n\to\infty} p_n = p$, such that $X_{p_n} \neq 0$ and $p_n \neq p$ for all $n \in \mathbb{N}$. Then

$$[X, Y]_{p_n} = \left.\frac{\partial}{\partial t}\right|_{t=0} (\varphi_t^* Y)_{p_n}$$

for all $n \in \mathbb{N}$. By continuity in the base point of both sides of the above expression we take their limit as $n \to \infty$ and conclude that $[X, Y]_p = \left.\frac{\partial}{\partial t}\right|_{t=0} (\varphi_t^* Y)_p$ as claimed □

Proposition 1.127 gives an answer to the question what the lie bracket of two vector fields should mean geometrically: $[X, Y]$ measures the infinitesimal change of Y along integral curves of X or, by skew-symmetry, the negative infinitesimal change of X along integral curves of Y, both via the pullback. This motivates the following definition:

Definition 1.128. The **Lie derivative** of a vector field $Y \in \mathfrak{X}(M)$ with respect to[16] $X \in \mathfrak{X}(M)$ is defined as

$$\mathcal{L}_X(Y) := [X, Y] \in \mathfrak{X}(M).$$

We will see in Section 2.2 that there is a different and very important important alternative concept how to measure infinitesimal changes of vector fields or, more general, sections of vector bundles. Next we will study how to obtain, in a natural way, new vector bundles from given bundles. This will allow us to define what a tensor field should be, which are central objects in every flavour of differential geometry and in applications in physics.

Definition 1.129. Let $\pi_E : E \to M$ be a vector bundle of rank k. The **dual vector bundle** $\pi_{E^*} : E^* \to M$ is pointwise given by

$$\pi_{E^*}^{-1}(p) E_p^* := \operatorname{Hom}_{\mathbb{R}}(E_p, \mathbb{R})$$

for all $p \in M$. The topology, smooth manifold structure, and bundle structure on E^* is obtained as follows. Let $\{(\psi_i, V_i) \mid i \in A\}$ be a collection of local trivializations of a vector bundle E of rank k, such that there exists an atlas $\mathcal{A} = \{(\varphi_i, \pi_E(V_i)) \mid i \in A\}$ of M.[17] Then $\mathcal{B} := \{((\varphi_i \times \mathrm{id}_{\mathbb{R}^k}) \circ \psi_i, V_i) \mid i \in A\}$ is an atlas on E. Recall that for any finite dimensional real vector space W, $(W^*)^*$ and W are isomorphic via

$$W \ni v \mapsto (\omega \mapsto \omega(v)), \ \omega \in W^*.$$

The topology on E^* is given by pre-images of open images of the **dual local trivializations** which are defined by

$$\widetilde{\psi_i} : \pi_{E^*}^{-1}(\pi_E(V_i)) \to \pi_E(V_i) \times \mathbb{R}^k, \quad \omega_p \mapsto (p, w),$$

[16] Or: "in direction of".

[17] Exercise: Show that such a choice is always possible.

where $w \in \mathbb{R}^k$ is the unique vector, such that $\omega_p(v_p) = \langle w, \mathrm{pr}_{\mathbb{R}^k}(\pi_E(v_p))\rangle$ for all $v_p \in \pi_E^{-1}(p)$ and $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product on $\mathbb{R}^k$ induced by its canonical coordinates. The **dual atlas** $\mathcal{B}^*$ on E^* is then defined by

$$\mathcal{B}^* := \{((\varphi_i \times \mathrm{id}_{\mathbb{R}^k}) \circ \widetilde{\psi_i}, V_i) \mid i \in A\}.$$

It follows that $E^* \to M$ is a vector bundle of rank k.

Exercise 1.130. Show that the transition functions of $E^* \to M$ fulfil

$$\widetilde{\psi_i} \circ \widetilde{\psi_j}^{-1} : (p, w) \mapsto \left(p, (A_p^{-1})^T w\right)$$

for all $p \in \pi_E(V_i)$, where $A : \pi_E(V_i) \to \mathrm{GL}(n)$ is given by the transition functions of $E \to M$,

$$\psi_i \circ \psi_j^{-1} : (p, v) \mapsto (p, A_p v).$$

Exercise 1.131. Show that $(E^*)^* \to M$ is isomorphic to $E \to M$ as a vector bundle for any vector bundle $E \to M$.

The most important example of a dual bundle is the dual to the tangent bundle of a smooth manifold (at least for this course).

Definition 1.132. The vector bundle $T^*M := (TM)^* \to M$ is called the **cotangent bundle** of M. Pointwise we denote $T_p^*M = (TM)_p^*$ for all $p \in M$. As for the tangent bundle we identify for any $U \subset M$ open and $p \in U$ the vector spaces $T_p^*U \cong T_p^*M$ via the inclusion map.

Similar to the tangent bundle, cf. Proposition 1.75, an atlas on M induces an atlas on the total space T^*M that is compatible with the bundle structure of T^*M as the dual bundle of TM. We will specify how a given local coordinate system $\varphi = (x^1, \ldots, x^n)$ on an open set $U \subset M$ induces a local coordinate system on the total space T^*M. Let $\pi_{T^*M} : T^*M \to M$ denote the projection. In the tangent bundle case we used equation (1.10) to define the induced local charts. This definition uses the action of tangent vectors $v \in T_pM$, $p \in U$, on the coordinate functions $(x^1, \ldots, x^n)$. In our present case of the cotangent bundle, elements in T_p^*M are linear maps

$$\omega \in T_p^*M, \quad \omega : v \mapsto \omega(v) \in \mathbb{R} \quad \forall v \in T_pM.$$

Thus we cannot let them act in any sensible way on the coordinate functions. However, recall that the coordinate functions induce a basis of T_pM for all $p \in U$, cf. Proposition 1.46. This motivates defining a local coordinate system on T^*M by

$$\widetilde{\psi} : \pi_{T^*M}^{-1}(U) \to \varphi(U) \times \mathbb{R}^n, \quad \widetilde{\psi} : \omega \mapsto \left(\varphi(\pi_{T^*M}(\omega)), \omega\left(\left.\frac{\partial}{\partial x^1}\right|_p\right), \ldots, \omega\left(\left.\frac{\partial}{\partial x^n}\right|_p\right)\right). \quad (1.13)$$

This local coordinate system is dual to the local coordinate system ψ on π_{TM}^{-1} as in equation (1.10) in the sense that for all $\omega_p \in T_p^*M$ and all $v_p \in T_pM$

$$\omega_p(v_p) = \langle \mathrm{pr}_{\mathbb{R}^n}(\widetilde{\psi}(\omega_p)), \mathrm{pr}_{\mathbb{R}^n}(\psi(v_p))\rangle, \quad (1.14)$$

where $\mathrm{pr}_{\mathbb{R}^n}$ denotes the canonical projection to the vector part and $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product induced by the canonical coordinates on $\mathbb{R}^n$. The independence of the chosen local coordinate system of the right hand side of (1.14) follows from Exercise 1.130.

We have defined vector fields as sections in the tangent bundle of a smooth manifold. Sections of the cotangent bundle are of the same importance as vector fields when studying smooth manifolds.

Definition 1.133. Sections in $T^*M \to M$ are called **1-forms** and denoted by $\Omega^1(M) := \Gamma(T^*M)$. For $U \subset M$ open, sections in $\Gamma(T^*M|_U)$ are denoted by $\Omega^1(U)$ and called **local 1-forms**.

A straightforward way of obtaining explicit examples of 1-forms is as follows.

Example 1.134. Let $f \in C^\infty(M)$. Then the **differential**[18] **of** f, $df \in \Omega^1(M)$, is given by

$$df : p \mapsto df_p.$$

In local coordinates $(x^1, \dots, x^n)$ we have $df\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial f}{\partial x^i}$ for all $1 \leq i \leq n$. This implies that df can locally be written as

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i.$$

In particular it follows for $f = x^j$ that the **coordinate 1-forms** dx^j fulfil $dx^j\left(\frac{\partial}{\partial x^i}\right) \equiv \delta_i^j$ on the domain of definition of the local coordinates. This in in accordance with the pointwise version of this statement Example 1.50.

Lemma 1.135. Let M be a smooth manifold and $\varphi = (x^1, \dots, x^n)$ be local coordinates defined on an open set $U \subset M$ and let $p \in U$ be arbitrary but fixed. Then

$$\{dx^i_p \mid 1 \leq i \leq n\}$$

is a basis of T_p^*M. It is precisely the dual basis to the basis $\left\{ \left.\frac{\partial}{\partial x^i}\right|_p \,\middle|\, 1 \leq i \leq n \right\}$ of T_pM. Any local 1-form $\omega \in \Omega^1(U)$ can be written as

$$\omega = \sum_{i=1}^n f_i dx^i \tag{1.15}$$

with uniquely determined smooth functions $f_i \in C^\infty(U)$ for $1 \leq i \leq n$.

Proof. The first two claims follow from Proposition 1.46 and Exercise 1.50. Next we observe that for any $f_i \in C^\infty(U)$, $1 \leq i \leq n$, the right hand side of equation (1.15) is a local section of T^*M[19] by the construction of the smooth manifold structure on the total space T^*M via charts of the form (1.13), which in particular implies that dx^i is a local 1-form. On the other hand for a given local 1-form ω, define

$$\omega_i := \omega\left(\frac{\partial}{\partial x^i}\right)$$

for all $1 \leq i \leq n$. It now suffices to show that $\omega_i \in C^\infty(U)$ and, after that, to define $f_i := \omega_i$. ω_i being a local smooth function follows from observing that by equation (1.14), $\omega_i \circ \varphi^{-1}$ is precisely the i-th entry in the vector part of $\widetilde{\psi} \circ \omega \circ \varphi^{-1}$ and thereby by definition a smooth map. Uniqueness of the f_i can be shown as follows. Suppose that locally

$$\omega = \sum_{i=1}^n f_i dx^i = \sum_{i=1}^n \widetilde{f}_i dx^i \tag{1.16}$$

such that for at least one $1 \leq j \leq n$, $f_j \neq \widetilde{f}_j$. Choose $p \in U$, such that $f_j(p) \neq \widetilde{f}_j(p)$. Then

$$\left(\sum_{i=1}^n f_i dx^i\right)\left(\left.\frac{\partial}{\partial x^j}\right|_p\right) = f_j(p) \neq \widetilde{f}_j(p) = \left(\sum_{i=1}^n \widetilde{f}_i dx^i\right)\left(\left.\frac{\partial}{\partial x^j}\right|_p\right)$$

which is a contradiction. □

[18] Cf. Definition 1.93.

[19] Recall that our definition of sections required them to be smooths maps.

In order to check whether a fibrewise map $\omega : M \to T^*M$ is a 1-form it suffices to check how it behaves when applied to vector fields and the converse statement also holds true:

Lemma 1.136. Let $\omega : M \to T^*M$, $\omega : p \mapsto T_p^*M$, be a fibrewise map. Then $\omega \in \Omega^1(M)$ if and only if for all $X \in \mathfrak{X}(M)$ the function $\omega(X) : p \mapsto \omega(X)(p)$ is smooth.

Proof. Exercise. [Hint: Use Lemma 1.135 and bump functions.] □

Example 1.134 and Lemma 1.135 motivate viewing the coordinate 1-forms as dual objects to coordinate vector fields. Indeed we obtain the following more abstract statement reinforcing this point of view.

Proposition 1.137. $\Omega^1(M)$ is isomorphic as a $C^\infty(M)$-module to the $C^\infty(M)$-module dual to $\mathfrak{X}(M)$, i.e.

$$\Omega^1(M) \cong \mathrm{Hom}_{C^\infty(M)}(\mathfrak{X}(M), C^\infty(M)).$$

Proof. Let $\alpha \in \Omega^1(M)$. we have seen in the proof of Lemma 1.135 that for any choice of local coordinates $(x^1, \ldots, x^n)$ on M, $\alpha\left(\frac{\partial}{\partial x^i}\right)$ is a local smooth function. Recall Proposition 1.88 and choose a locally finite countable partition of unity $\{b_i : U_i \to [0,1] \mid i \in I\}$ subordinate to a countable atlas $\{(\varphi_i = (x_i^1, \ldots, x_i^n), U_i) \mid i \in I\}$ of M. Write $X(p) = \sum_{i\in I} b_i(p)X(p)$, observe that this sum is finite for all fixed $p \in M$ and that $b_iX \in \mathfrak{X}(U_i)$ for all $i \in I$. We can write b_iX in local coordinates as

$$b_iX = \sum_{j=1}^n b_iX(x_i^j)\frac{\partial}{\partial x_i^j} =: \sum_{j=1}^n b_iX_i^j\frac{\partial}{\partial x_i^j}$$

and observe that, since all b_i, $i \in I$, are in particular bump functions, $b_iX_i^j \in C^\infty(U_i)$ can be trivially extended to a smooth function on M and that $b_iX \in \mathfrak{X}(U_i)$ can be trivially extended to be a vector field on the whole manifold M for all $i \in I$ and all $1 \leq j \leq n$.

$$A_\alpha(X) = \alpha(X) = \sum_{i\in I} \alpha(b_iX) = \sum_{i\in I}\sum_{j=1}^n b_i\alpha_jX_i^j. \tag{1.17}$$

The above sum on the right hand side is a locally finite sum of bump functions (defined on the respective U_i which we trivially extent to M). This means that for all $p \in M$ fixed there exists an open neighbourhood $U \subset M$ of p, such that the set

$$\{(i,j) \in I \times \{1, \ldots, n\} \mid b_i\alpha_jX_i^j(q) \neq 0 \text{ for at least one } q \in U\}$$

is finite. Hence, the right hand side of (1.17) is indeed a smooth function defined on M since it is, locally, the sum of finitely many smooth functions. This shows α defines a well-defined $C^\infty(M)$-linear map

$$A_\alpha : \mathfrak{X}(M) \to C^\infty(M), \quad X \mapsto \alpha(X),\ A(X)(p) := \alpha_p(X_p)\ \forall p \in M.$$

On the other hand let $A : \mathfrak{X}(M) \to C^\infty(M)$ be a $C^\infty(M)$-linear map. For a given $A \in \mathrm{Hom}_{C^\infty(M)}(\mathfrak{X}(M), C^\infty(M))$ define a fibre-preserving map

$$\alpha_A : M \to T^*M, \quad \alpha_A|_p(v) := A(X)(p)$$

for $X \in \mathfrak{X}(M)$ with $X_p = v$. We need to show that this definition does not depend on the choice of X and that α_A is in fact a smooth map. Since $A(X+Y)(p) = A(X)(p) + A(Y)(p)$ for all $X, Y \in \mathfrak{X}(M)$ and all fixed $p \in M$ it suffices to show that $A(X)(p) = 0$ for all $X \in \mathfrak{X}(M)$ with

$X_p = 0$. Let $(x^1, \dots, x^n)$ be local coordinates on $U \subset M$ with $p \in U$ so that X with $X_p = 0$ is locally of the form

$$X = \sum_{i=1}^n X^i \frac{\partial}{\partial x^i},$$

$X^i \in C^\infty(U)$ for all $1 \leq i \leq n$. Then $X_p = 0$ precisely means that $X^i(p) = 0$ for all $1 \leq i \leq n$. Since $A(fX)(p) = f(p)A(X)(p)$ for all $f \in C^\infty(M)$, we can choose a bump function $b \in C^\infty(M)$, so that $\operatorname{supp}(b) \subset U$ is compactly embedded and such that there exists a compactly embedded set $V \subset U$ with non-empty interior containing p and $b|_V \equiv 1$. Then for all $1 \leq i \leq n$, bX^i are also bump functions on $U \subset M$ and thus can be smoothly extended to M (note that the X^i might have "bad" behaviour when approaching ∂U, e.g. do not converge). Furthermore, $b\frac{\partial}{\partial x^i}$ can be extended to a globally defined vector field on M for all $1 \leq i \leq n$ by setting

$$b\frac{\partial}{\partial x^i}\bigg| q = 0$$

for all $q \in M \setminus U$. We will for simplicity write $bX^i \in C^\infty(M)$ and $b\frac{\partial}{\partial x^i} \in \mathfrak{X}(M)$ for all $1 \leq i \leq n$. We now calculate

$$\begin{aligned} A(X)(p) &= b^2(p)A(X)(p) = A(b^2X)(p) = A\left(\sum_{i=1}^n (bX^i)\left(b\frac{\partial}{\partial x^i}\right)\right)(p) \\ &= \left(\sum_{i=1}^n bX^i A\left(b\frac{\partial}{\partial x^i}\right)\right)(p) = \sum_{i=1}^n X^i(p) A\left(b\frac{\partial}{\partial x^i}\right)(p) = 0. \end{aligned}$$

It remains to show that α_A is smooth. This follows with the help of Lemma 1.136, the above result, and a similar construction using a locally finite partition of unity as for the other direction of the proof. One can further check that $A_{\alpha_A} = A$ and $\alpha_{A_\alpha} = \alpha$, that is that the two constructions are inverse to each other. □

Exercise 1.138. Work out the missing details of the "$\supset$" direction in the proof of Proposition 1.137.

As for vector fields, cf. Definition 1.119, we can define the pullback and pushforward of 1-forms under diffeomorphisms. Be wary of the differences!

Definition 1.139. Let $F : M \to N$ be a diffeomorphism and let $\alpha \in \Omega^1(M)$, $\beta \in \Omega^1(N)$. The **pushforward** of α under F is the 1-form $F_*\alpha \in \Omega^1(N)$ given by

$$(F_*\alpha)_q := \alpha_{F^{-1}(q)} \circ d(F^{-1})_q \quad \forall q \in N.$$

The **pullback** of β under F is the 1-form $F^*\beta \in \Omega^1(M)$ given by

$$(F^*\beta)_p := \beta_{F(p)} \circ dF_p \quad \forall p \in M.$$

The above compositions denote compositions of linear maps.

Remark 1.140. The pullback of a 1-form $\beta \in \Omega^1(N)$ is well-defined for any smooth map $F : M \to N$.

Next we will study some constructions on how to obtain new vector bundles from given vector bundles.

Definition 1.141. Let $\pi_E : E \to M$ be a vector bundle of rank k and $\pi_F : F \to M$ a vector bundle of rank ℓ over an n-dimensional smooth manifold M. The **Whitney**[20] **sum of E and F** is the the direct sum of the two vector bundles $\pi_{E\oplus F} : E \oplus F \to M$ with fibres

$$(E \oplus F)_p = \pi_{E\oplus F}^{-1}(p) := E_p \oplus F_p.$$

The structure of a vector bundle on $E \oplus F = \bigsqcup_{p\in M} (E_p \oplus F_p)$ is then explained by Proposition 1.73 and the requirement that the following maps are local trivializations of $E \oplus F$. Let $\{(\psi_i^E, V_i^E) \mid i \in I\}$ and $\{(\psi_i^F, V_i^F) \mid i \in I\}$ be coverings of local trivializations of E and F, respectively, such that $U_i := \pi_E(V_i^E) = \pi_F(V_i^F)$ for all $i \in I$ and such that there exists an atlas $\mathcal{A} = \{(\varphi_i, U_i) \mid i \in I\}$ of M. We require now require that with

$$\phi_i^{-1} := (\psi_i^E \oplus \psi_i^F)^{-1} \circ (\Delta_M \times \mathbb{R}^{k+\ell}) : U_i \times \mathbb{R}^{k+\ell} \cong U_i \times (\mathbb{R}^k \times \mathbb{R}^\ell) \to \bigsqcup_{p\in U_i} (E_p \oplus F_p),$$
$$(p, v, w) \mapsto (\psi_i^E)^{-1}(p, v) \oplus (\psi_i^F)^{-1}(p, w) \quad \forall p \in U_i,\ v \in \mathbb{R}^k,\ w \in \mathbb{R}^\ell, \tag{1.18}$$

where $\Delta_M : p \mapsto (p, p) \in M \times M$ denotes the diagonal embedding and

$$\mathbb{R}^k \times \mathbb{R}^\ell \ni (v, w) \mapsto \begin{pmatrix} v \\ w \end{pmatrix} \in \mathbb{R}^{k+\ell}$$

the linear isomorphism, all ϕ_i, $i \in I$, are inverses of local trivializations covering $E \oplus F$. In order to use Proposition 1.73 we need to check that the transition functions have the required form. We obtain that for all $i, j \in I$, such that $U_i \cap U_j \neq \emptyset$,

$$\phi_i \circ \phi_j^{-1}(p, v, w) = (p, \tau_{ij}^E(p)v, \tau_{ij}^F(p)w),$$

where τ_{ij}^E and τ_{ij}^F are the transition functions of the local trivializations of E and F, respectively. Lastly, we simply need to define

$$\tau_{ij}^{E\oplus F}(p) := \left(\begin{array}{c|c} \tau_{ij}^E(p) & 0 \\ \hline 0 & \tau_{ij}^F(p) \end{array}\right) \in \mathrm{GL}(k + \ell)$$

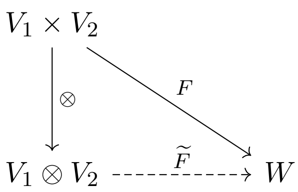
so that we can write $\phi_i \circ \phi_j^{-1}\left(p, \left(\begin{smallmatrix} v \\ w \end{smallmatrix}\right)\right) = \left(p, \tau_{ij}^{E\oplus F}(p) \left(\begin{smallmatrix} v \\ w \end{smallmatrix}\right)\right)$. Now all requirements in Proposition 1.73 are fulfilled and we conclude that $E \oplus F \to M$ is, indeed, a vector bundle of rank $k + \ell$.

Exercise 1.142. Show that the vector bundles $T(M \times N)$ and

$$TM \oplus TN \to M \times N, \quad (v_p, w_q) \mapsto (p, q) \quad \forall v_p \in T_pM\ \forall w_q \in T_qN,$$

are isomorphic as vector bundles for any two smooth manifolds M and N.

A construction similar to the Whitney sum is the tensor product of vector bundles. Recall that the tensor product of two real vector spaces V_1 of dimension n and V_2 of dimension m is a real vector space $V_1 \otimes V_2$ together with a bilinear map $\otimes : V_1 \times V_2 \to V_1 \otimes V_2$, such that for every real vector space W and every bilinear map $F : V_1 \times V_2 \to W$, there exist a unique linear map $\widetilde{F} : V_1 \otimes V_2 \to W$ making the diagram



[20] Hassler Whitney (1907 – 1989)

commute. The dimension of $V_1 \otimes V_2$ is $n \cdot m$. If $\{v_1^1, \dots, v_1^n\}$ and $\{v_2^1, \dots, v_2^m\}$ are a basis of V_1 and V_2, respectively, we can construct a choice of basis for $V_1 \otimes V_2$ explicitly. A basis of $V_1 \otimes V_2$ is given by $\{v_1^i \otimes v_2^j,\ 1 \leq i \leq n,\ 1 \leq j \leq m\}$, and $\widetilde{F}$ for a bilinear map F as above is given by

$$\widetilde{F} : v_1^i \otimes v_2^j \mapsto F(v_1^i, v_2^j)$$

on the basis vectors. By considering "$\otimes$" itself as a bilinear map from $V_1 \times V_2$ to $W = V_1 \otimes V_2$, we define[21] $v \otimes w$ for $v = \sum_{i=1}^{n} v^i v_1^i$, $w = \sum_{j=1}^{m} w^j v_2^i$, as

$$v \otimes w := \sum_{i=1}^{n} \sum_{j=1}^{m} v^i w^j \cdot v_1^i \otimes v_2^j.$$

An element $v \in V_1 \otimes V_2$ is called a **pure tensor** if it can be written as $v = v_1 \otimes v_2$ for some $v_1 \in V_1$ and $v_2 \in V_2$. In order to describe any linear map $L : V_1 \otimes V_2 \to W$ is suffices to know how it acts on pure tensors [Exercise: Prove the last statement.]

A very important example that you should keep in mind is the tensor product of a real vector space V with its dual, that is $V \otimes V^*$.

Exercise 1.143.

(i) Show that the real vector space of endomorphisms $\mathrm{End}(V)$ and $V \otimes V^*$ are isomorphic as real vector spaces via

$$V \otimes V^* \ni v \otimes \omega \mapsto (u \mapsto \omega(u)v) \in \mathrm{End}(V).$$

(ii) Show that $V \otimes \mathbb{R} \cong V$ and $V \otimes W \cong W \otimes V$ for all real vector spaces V and W.

For the **evaluation map**

$$\mathrm{ev} : V \times V^* \to \mathbb{R}, \quad (v, \omega) \mapsto \omega(v) \quad \forall v \in V,\ \omega \in V^*,$$

the induced map $\widetilde{\mathrm{ev}} : V \otimes V^* \to \mathbb{R}$ is called **contraction**. By saying that we contract $v \otimes \omega$ we simply mean sending it to $\omega(v)$. Further recall that $V_1 \otimes (V_2 \otimes V_3)$ and $(V_1 \otimes V_2) \otimes V_3$ are isomorphic. In the following we will deal with objects that, pointwise, are elements of vector spaces of the form

$$\underbrace{V \otimes \ldots \otimes V}_{r \text{ times}} \otimes \underbrace{V^* \otimes \ldots \otimes V^*}_{s \text{ times}}.$$

A contraction of an element $v_1 \otimes \ldots \otimes v_r \otimes \omega_1 \otimes \ldots \otimes \omega_s \in V \otimes \ldots \otimes V \otimes V^* \otimes \ldots \otimes V^*$ will stand for a map of the form

$$v_1 \otimes \ldots \otimes v_r \otimes \omega_1 \otimes \ldots \otimes \omega_s \mapsto \omega_\beta(v_\alpha) \cdot v_1 \otimes \ldots \widehat{\otimes v_\alpha} \otimes \ldots v_r \otimes \omega_1 \otimes \ldots \widehat{\otimes \omega_\beta} \otimes \ldots \omega_s \quad (1.19)$$

for $1 \leq \alpha \leq r$ and $1 \leq \beta \leq s$ fixed, where "$\widehat{\ }$" means that the element is supposed to be left out. This is precisely the induced map for the evaluation map in the (α, β)-th entry. If α and β are not further specified, any statement that contains such a contraction is supposed to hold for all possibilities of α and β.

We will now generalize the definition of a tensor product of vector spaces to vector bundles. Pointwise, the two definitions coincide.

[21] Make sure to understand why this is consistent with the definition of the tensor product.

Definition 1.144. Let $\pi_E : E \to M$ be a vector bundle of rank k and $\pi_F : F \to M$ be a vector bundle of rank ℓ and, as in Definition 1.141, let ψ_i^E and ψ_i^F, $i \in I$, be local trivializations of E and F, respectively, and $\mathcal{A}$ a fitting atlas of M with charts (φ_i, U_i), $i \in I$. The **tensor product of vector bundles** of E and F, $\pi_{E\otimes F} : E \otimes F \to M$, is the vector bundle given pointwise by

$$(E \otimes F)_p = \pi_{E\otimes F}^{-1}(p) := E_p \otimes F_p,$$

so that $E \otimes F := \bigsqcup_{p\in M} E_p \otimes F_p$. As in the construction of the Whitney sum of vector bundles, it suffices by Proposition 1.73 to construct local trivializations $\phi_i : \pi^{-1}(U_i) \to U_i\times\mathbb{R}^k\otimes\mathbb{R}^\ell \cong U_i\times\mathbb{R}^{k\ell}$ covering $E \otimes F$ with smooth vector parts of their transition functions in order to show that $E \otimes F$ is in fact a vector bundle. Analogous to equation (1.18) we set

$$\begin{aligned} \phi_i^{-1} &:= (\psi_i^E \otimes \psi_i^F)^{-1} \circ (\Delta_M \times \mathrm{id}_{\mathbb{R}^{k\ell}}) : U_i \times \mathbb{R}^{k\ell} \cong U_i \times (\mathbb{R}^k \otimes \mathbb{R}^\ell) \to \bigsqcup_{p\in U_i} (E_p \otimes F_p), \\ (p, v \otimes w) &\mapsto (\psi_i^E)^{-1}(p, v) \otimes (\psi_i^F)^{-1}(p, w) \quad \forall p \in U_i,\ v \in \mathbb{R}^k,\ w \in \mathbb{R}^\ell, \end{aligned} \tag{1.20}$$

where $\Delta_M : p \mapsto (p, p) \in M \times M$ again denotes the diagonal embedding and ϕ_i^{-1} on non-pure tensors is defined by linear extension for any $p \in U_i$ fixed. For the transition functions of the vector part in the change of local trivializations of $E \otimes F \to M$ we obtain for all $i, j \in I$, such that $U_i \cap U_j \neq \emptyset$,

$$\phi_i \circ \phi_j^{-1}(p, v \otimes w) = (p, \tau_{ij}^E(p)v \otimes \tau_{ij}^F(p)w),$$

where τ_{ij}^E and τ_{ij}^F are the transition functions of the local trivializations of E and F, respectively. We check [Exercise!] that the linear extension of

$$\mathbb{R}^k \otimes \mathbb{R}^\ell \ni v \otimes w \mapsto \tau_{ij}^E(p)v \otimes \tau_{ij}^F(p)w \in \mathbb{R}^k \otimes \mathbb{R}^\ell$$

actually is an invertible linear map and conclude with Proposition 1.73 that $E \otimes F \to M$ is indeed a vector bundle of rank $k\ell$.

Exercise 1.145. The endomorphism bundle of a vector bundle $E \to M$ is defined as

$$\mathrm{End}(E) := E \otimes E^* \to M.$$

Describe the transition functions of $\mathrm{End}(E) \to M$ induced by given transition functions on $E \to M$.

Definition 1.146. Let M be a smooth manifold and let $(r, s) \in \mathbb{N}_0 \times \mathbb{N}_0$ so that $r + s > 0$. The vector bundle

$$T^{r,s}M := \underbrace{TM \otimes \ldots \otimes TM}_{r \text{ times}} \otimes \underbrace{T^*M \otimes \ldots \otimes T^*M}_{s \text{ times}} \to M$$

is called the **bundle of** (r,s)**-tensors** of M. In this notation, $T^{1,0}M = TM$ and $T^{0,1}M = T^*M$. The (local) sections in the bundle of (r,s)-tensors are called **(local)** (r,s)**-tensor fields**, or simply **tensor fields** if (r,s) is clear from the context, and are denoted by

$$\mathfrak{T}^{r,s}(M) := \Gamma(T^{r,s}M).$$

In local coordinates $(x^1, \ldots, x^n)$ on $U \subset M$, tensor fields $A \in \mathfrak{T}^{r,s}(M)$ are of the form

$$\begin{aligned} A &= \sum_{\substack{1\le i_1,\ldots,i_r\le n \\ 1\le j_1,\ldots,j_r\le n}} A^{i_1\ldots i_r}{}_{j_1\ldots j_s} \frac{\partial}{\partial x^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_s}, \\ A^{i_1\ldots i_r}{}_{j_1\ldots j_s} &\in C^\infty(U) \quad \forall 1 \le i_1, \ldots, i_r, j_1, \ldots, j_s \le n. \end{aligned} \tag{1.21}$$

The above local form of tensor fields is commonly called **index notation of tensor fields**. This is justified by the fact that locally A is uniquely determined by the local smooth functions ${A^{i_1\dots i_r}}_{j_1\dots j_s}$ on chart neighbourhoods of an atlas of M. Note that the summation in (1.21) "pairs up" coinciding upper and lower indices. In physics literature, the summation signs are usually omitted, which is called the **Einstein summation convention**. We will not be using that convention a.k.a. notation but instead leave out the ranges of the summations from here on whenever they are clear from the context. For example, a vector field $X \in \mathfrak{X}(M)$ on an n-dimensional smooth manifold M will then locally be written as

$$X = \sum X^i \frac{\partial}{\partial x^i}.$$

If $A \in \mathcal{T}^{r,s}(M)$ with $r > 0$ and $s > 0$ we can **contract** A in the i,j-th index, $1 \leq i \leq r$, $1 \leq j \leq s$, which is pointwise in local coordinates defined as in (1.19), and obtain a tensor field in $\mathcal{T}^{r-1,s-1}(M)$.[22]

Remark 1.147. Recall Proposition 1.137 and the construction of the tensor product via its universal property. One can show that $\mathcal{T}^{r,s}(M)$ is as $C^\infty(M)$ module isomorphic to the $C^\infty(M)$-multilinear maps $\mathrm{Hom}_{C^\infty(M)}(\Omega^1(M)^{\times r} \times \mathfrak{X}(M)^{\times s}, C^\infty(M))$. The proof needs some knowledge about tensor products of modules, but essentially works as the proof of Proposition 1.137.

Exercise 1.148.

(i) Work out the transformation laws for (r,s)-tensor fields when changing coordinates. As an example consider linear (global) change of coordinates in $\mathbb{R}^n$, i.e.

$$\begin{pmatrix} u^1 \\ \vdots \\ u^n \end{pmatrix} = B \begin{pmatrix} w^1 \\ \vdots \\ w^n \end{pmatrix}$$

for $(u^1,\dots,u^n)$ the canonical coordinates and $B \in \mathrm{GL}(n)$. [Even though it is a bit tedious, do **not** skip this exercise!]

(ii) Show that contraction of tensor fields is well-defined, i.e. show that for a tensor field in local coordinates first contracting and then changing coordinates yields the same expression as first changing coordinates and then contracting. This in particular means that the contraction of an **endomorphism field** in $\mathcal{T}^{1,1}(M)$ is a well-defined smooth function on M.

(iii) Check that the contraction of an endomorphism field A is pointwise in local coordinates precisely the trace of the endomorphism $A_p : T_pM \to T_pM$.

We know what it means to transport vector fields and 1-forms via diffeomorphisms from one smooth manifold to another. For 1-forms we have seen that we can pull them back with respect to any smooth map, not just diffeomorphisms. These constructions work for general tensor fields as well by applying them entry-wise. Observe the following:

Remark 1.149. For any $a \in \mathcal{T}^{r,s}(M)$, $b \in \mathcal{T}^{R,0}(M)$, $c \in \mathcal{T}^{0,S}(M)$,

$$b \otimes a \in \mathcal{T}^{r+R,s}(M), \quad a \otimes c \in \mathcal{T}^{r,s+S}(M),$$

where the tensor product is understood over $C^\infty(M)$.[23] For the above reason we **identify** $C^\infty(M)$ **with** $\mathcal{T}^{0,0}(M)$ so that a $(0,0)$-tensor field is simply a smooth function. For $f \in C^\infty(M)$, we set $f \otimes a := fa$ for all $a \in \mathcal{T}^{r,s}(M)$.

[22] $\mathcal{T}^{0,0}(M) := C^\infty(M)$, see Remark 1.149.

[23] This means that $(fb) \otimes a = b \otimes (fa)$ for all $f \in C^\infty(M)$, the construction of the tensor product of modules is analogous to the construction of tensor products of vector spaces. For a reference see e.g. [].

Definition 1.150. Let M, N be smooth manifolds and let $F : M \to N$ be a diffeomorphism. The **pushforward and pullback of tensor fields** under F are the unique $\mathbb{R}$-linear maps

$$F_* : \mathcal{T}^{r,s}(M) \to \mathcal{T}^{r,s}(N),$$
$$F^* : \mathcal{T}^{r,s}(N) \to \mathcal{T}^{r,s}(M),$$

such that

(i) $F_* : \mathcal{T}^{1,0}(M) \to \mathcal{T}^{1,0}(N)$ is the pushforward of vector fields, $F^* : \mathcal{T}^{1,0}(N) \to \mathcal{T}^{1,0}(M)$ is the pullback of vector fields,

(ii) $F_* : \mathcal{T}^{0,1}(M) \to \mathcal{T}^{0,1}(N)$ is the pushforward of 1-forms, $F^* : \mathcal{T}^{0,1}(N) \to \mathcal{T}^{0,1}(M)$ is the pullback of 1-forms,

(iii) $F_*(b \otimes a) = (F_*b) \otimes (F_*a)$ and $F^*(b \otimes a) = (F^*b) \otimes (F^*a)$ for all $a \in \mathcal{T}^{r,s}(M)$, $b \in \mathcal{T}^{R,0}(M)$,

(iv) $F_*(a \otimes c) = (F_*a) \otimes (F_*c)$ and $F^*(a \otimes c) = (F^*a) \otimes (F^*c)$ for all $a \in \mathcal{T}^{r,s}(M)$, $c \in \mathcal{T}^{0,S}(M)$.

For $f \in C^\infty(M)$, $g \in C^\infty(N)$, we set

$$F_*(f) := f \circ F^{-1}, \quad F^*g := g \circ F$$

so that $F_*(fa) = F_*(f)F_*(a)$ and $F^*(gb) = F^*(g)F^*(b)$ for all $f \in C^\infty(M)$, $g \in C^\infty(N)$, $a \in \mathcal{T}^{r,s}(M)$, $b \in \mathcal{T}^{r,s}(N)$.

The above definition might look worse than it actually is. If we are given some specific tensor field, say, an endomorphism field $A \in \mathcal{T}^{1,1}(M)$ which is in local coordinates on $U \subset M$ of the form

$$A = \sum A^i{}_j \frac{\partial}{\partial x^i} \otimes dx^j$$

and a diffeomorphism $F : M \to N$, all we need to do to calculate the local form of F_*A is to choose fitting local coordinates on (or on a subset of) $F(U) \subset N$, and after possibly shrinking U calculate $F_*\left(\frac{\partial}{\partial x^i}\right)$, $F_*(dx^j)$, for all $1 \leq i, j \leq n$. Then we can use the $\mathbb{R}$-linearity of the tensor product to get a local form of $F_*(A)$.

Remark 1.151. The pullback of $(0, s)$-tensors on N is well-defined even if $F : M \to N$ is not a diffeomorphism.

Lemma 1.152. Contraction of tensor fields commute with the pushforward and with the pullback defined above.

Proof. By the $\mathbb{R}$-linearity and Definition 1.150, (i) & (ii), it suffices to show that

$$F_*(\alpha(X)) = F_*(\alpha)(F_*(X))$$

and

$$F^*(\beta(Y)) = F^*(\beta)(F^*(Y))$$

for all diffeomorphisms $F : M \to N$ and all $\alpha \in \Omega^1(M)$, $\beta \in \Omega^1(N)$, $X \in \mathfrak{X}(M)$, $Y \in \mathfrak{X}(N)$. This follows directly from Definitions 1.119 & 1.139. □

Exercise 1.153.

(i) Show that the pushforward w.r.t. a diffeomorphism $F : M \to N$ is inverse to the pullback w.r.t. the inverse of the diffeomorphism $F^{-1} : N \to M$, independently of the type of tensor fields.

(ii) Determine all vector fields on S^1 that are invariant under the pushforward of all rotations in the ambient space $\mathbb{R}^2$ restricted to S^1. $X \in \mathfrak{X}(S^1)$ being invariant under $F : S^1 \to S^1$ means that $X_p = (F_*X)_p$ for all $p \in S^1$.

Recall the definition of the Lie derivative of vector fields. Geometrically, the Lie derivative is one way of measuring the infinitesimal change of a vector field along the local flow of another vector field. We can, analogously, define the Lie derivative of general tensor fields with respect

Definition 1.154. Let M be a smooth manifold, $X \in \mathfrak{X}(M)$ a vector field, and $A \in \mathcal{T}^{r,s}(M)$ a tensor field. Then the **Lie derivative of A in direction of X**, $\mathcal{L}_X A \in \mathcal{T}^{r,s}(M)$, is defined as

$$(\mathcal{L}_X A)_p := \frac{\partial}{\partial t}\bigg|_{t=0} (\varphi_t^* A)_p \quad \forall p \in M,$$

where $\varphi : I \times U \to M$ is any local flow of X near $p \in M$.

Note that $(\varphi_t^* A)_p$ is, for p fixed, for all $t \in I$ contained in the same vector space $T_pM \otimes \ldots \otimes T_pM \otimes T_p^*M \otimes \ldots \otimes T_p^*M$, thus $\mathcal{L}_X A$ is well-defined.

Remark 1.155. The above definition is consistent with the identification $\mathcal{T}^{0,0}(M) = C^\infty(M)$.

Proposition 1.156. The Lie derivative of tensor fields is a **tensor derivation**, i.e. it is compatible with all possible contractions and fulfils the Leibniz rule

$$\mathcal{L}_X(A \otimes B) = \mathcal{L}_X A \otimes B + A \otimes \mathcal{L}_X B$$

for all vector fields X and all tensor fields A, B, such that $A \otimes B$ is defined.

Proof. To show compatibility with contractions it suffices to show that it holds true for an endomorphism field $A \in \mathcal{T}^{1,1}(M)$. All other possible cases will follow by induction and the Leibniz rule. We will prove first that the Leibniz rule is fulfilled. Let $p \in M$ be fixed and A, B, two tensor fields, such that $A \otimes B$ is defined. First assume that $(A \otimes B)_p = A_p \otimes B_p \neq 0$. Let $X \in \mathfrak{X}(M)$ be arbitrary but fixed and denote by $\varphi : I \times U \to M$ its local flow near p with $U \subset M$ contained in a chart neighbourhood for some local coordinates. We can find an interval $(-\varepsilon, \varepsilon) \subset I$ for $\varepsilon > 0$ small enough, such that in the local coordinates on U and the induced coordinates on the fitting (r, s)-tensor bundles ψ and ϕ, the pullbacks of A and B w.r.t. the local flow of X are of the form

$$\psi((\varphi_t^* A)_p) = (p, a(t)v), \quad \phi((\varphi_t^* B)_p) = (p, b(t)w) \quad \forall t \in (-\varepsilon, \varepsilon).$$

In the above equation, $0 \neq v \in \mathbb{R}^{N_1}$ and $0 \neq w \in \mathbb{R}^{N_2}$ are fixed nonzero vectors and N_1, N_2, depend on the type of tensor field that A and B are. The expressions $a(t)$ and $b(t)$ stand for smooth and uniquely defined maps

$$a : (-\varepsilon, \varepsilon) \to \mathrm{GL}(N_1), \quad b : (-\varepsilon, \varepsilon) \to \mathrm{GL}(N_2),$$

with $a(0) = \mathrm{id}_{\mathbb{R}^{N_1}}$ and $b(0) = \mathrm{id}_{\mathbb{R}^{N_2}}$. Thus, in order to prove the Leibniz property, it suffices to show that for any finite dimensional real vector spaces V, $\dim(V) = N_1$, and W, $\dim(W) = N_2$, and any smooth maps a and b as above,

$$\frac{\partial}{\partial t}\bigg|_{t=0} ((a(t)v) \otimes (b(t)w)) = (a'(0)v) \otimes w + v \otimes (b'(0)) \tag{1.22}$$

for all $v \in V$, $w \in W$. This follows from the defining universal property of the tensor product of vector spaces as follows. Let $L : V \times W \to \mathbb{R}$ be any bilinear map and $\widetilde{L} : V \otimes W \to \mathbb{R}$ the corresponding linear map, so that $L(a(t)v, b(t)w) = \widetilde{L}((a(t)v) \otimes (b(t)w))$ for all $v \in V$, $w \in W$, $t \in (-\varepsilon, \epsilon)$. By taking the t-derivative at $t = 0$ on both sides we obtain

$$\left.\frac{\partial}{\partial t}\right|_{t=0} \widetilde{L}((a(t)v) \otimes (b(t)w)) = \widetilde{L}((a'(0)v) \otimes w + v \otimes (b'(0)w)).$$

Since L and thus $\widetilde{L}$ were arbitrary, the above statement hold in particular for all component functions. This shows (1.22) and, hence, proves the Leibniz property. To obtain the compatibility with contractions it is enough to consider $V = W^*$ and $L = \mathrm{ev}$ the evaluation map. Then $\widetilde{L}$ is precisely the contraction.

Next assume that $(A \otimes B)_p = 0$ and that there exists a convergent sequence $\{p_n\}_{n\in\mathbb{N}}$ with $p_n \to p$ as $n \to \infty$, such that $(A \otimes B)_{p_n} \neq 0$ for all $n \in N$. Then the statement of this proposition follows with a continuity argument similar to the one used in Proposition 1.127.

Lastly assume that $(A \otimes B)_p = 0$ and $A \otimes B$ vanishes identically on an open neighbourhood $U \subset M$ of p. Then A or B must already vanish identically on U. Without loss of generality we can assume that U is a chart neighbourhood, choose a fitting bump function b with $\mathrm{supp}(b) \subset U$ compactly embedded, so that the locally defined prefactors in the local forms of A and B, multiplied with said bump function, are globally defined smooth functions. Now we use that bA and bB vanish identically and in some smaller open neighbourhood $V \subset U$ coincide with A and B, respectively. Thus on V if $bA \equiv 0$ we obtain $\mathcal{L}_X(A) = \mathcal{L}_X(bA) = \mathcal{L}_X(0) = 0$ and a similar identity for B and $A \otimes B$. This finishes the proof. □

Corollary 1.157. $(\mathcal{L}_X\alpha)(Y) = X(\alpha(Y)) - \alpha([X, Y])$ for all $X, Y \in \mathfrak{X}(M)$ and all $\alpha \in \Omega^1(M)$.

Exercise 1.158. Show that $\mathcal{L}_X(df) = d(\mathcal{L}_X f)$ for all $f \in C^\infty(M)$, $X \in \mathfrak{X}(M)$.

2 Pseudo-Riemannian metrics, connections, and geodesics

2.1 Pseudo-Riemannian metrics and isometries

We start this section with quickly recalling some facts from linear algebra on (finite dimensional) vector spaces equipped with a scalar product, that is a symmetric bilinear map with values in $\mathbb{R}$.

Remark 2.1. Let V be a finite-dimensional real vector space. A **pseudo-Euclidean scalar product on** V is a nondegenerate symmetric bilinear map

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}.$$

Nondegenerate means that there exists no proper linear subspace $W \subset V$, such that $\langle \cdot, \cdot \rangle|_{W\times V} \equiv 0$. The **index of** $\langle \cdot, \cdot \rangle$ is defined as the number of its negative eigenvalues when written as a symmetric $\dim(V) \times \dim(V)$-matrix. The index does not depend on the choice of basis of V, this is **Sylvester's law of inertia**. If the index of the pseudo-Euclidean scalar product is zero, it is simply called **Euclidean scalar product**. A vector space equipped with a (pseudo)-Euclidean scalar product is called **(pseudo)-Euclidean vector space**. Prominent examples are $\mathbb{R}^n$ together with the Euclidean scalar product that is given by the dot-product, i.e.

$$\langle v, w \rangle = \sum_{i=1}^{n} v^i w^i,$$

and $\mathbb{R}^{n+1}$ together with the **Minkowski scalar product**

$$\langle v, w \rangle = -v^{n+1}w^{n+1} + \sum_{i=1}^{n} v^i w^i.$$

Note that in certain fields of theoretical physics one uses an overall sign in front of the Minkowski scalar product. The **length** of a vector $v \in V$ with respect to a pseudo-Euclidean scalar product $\langle \cdot, \cdot \rangle$ is defined as

$$\|v\| := \sqrt{|\langle v, v \rangle|}.$$

If $\langle \cdot, \cdot \rangle$ is a pseudo-Euclidean scalar product with negative and positive eigenvalues of the representation matrix, one says that a vector v is **spacelike** if $\langle v, v \rangle > 0$, **timelike** if $\langle v, v \rangle < 0$, and **null** if $\langle v, v \rangle = 0$. If $\langle \cdot, \cdot \rangle$ is Euclidean each nonzero vector has positive length. Let $A \in \mathrm{GL}(V)$ describe a change of basis in a pseudo-Euclidean vector space $(V, \langle \cdot, \cdot \rangle)$ in the sense that A maps the new basis to the given one and assume that the representation matrix of $\langle \cdot, \cdot \rangle$ is given by the symmetric matrix B. Then in the new basis, the representation matrix of $\langle \cdot, \cdot \rangle$ is given by $A^T B A$.[24] Two pseudo-Euclidean vector spaces $(V, \langle \cdot, \cdot \rangle_V)$ and $(W, \langle \cdot, \cdot \rangle_W)$ are called **isometric** if there exist a linear isomorphism $A : V \to W$, such that $\langle \cdot, \cdot \rangle_V = \langle A \cdot, A \cdot \rangle_W$. A is then called **(linear) isometry**. Two finite-dimensional pseudo-Euclidean vector spaces are isometric if and only if their dimension and index of the scalar product coincide. Note that any pseudo-Euclidean scalar product might be interpreted as an element in $\mathrm{Sym}^2(V^*)$ which denotes the set of symmetric two-tensors in $V^* \otimes V^*$.

Exercise 2.2. Show that for any pseudo-Euclidean vector space $(V, \langle \cdot, \cdot \rangle)$, $\langle \cdot, \cdot \rangle$ is completely determined by its value on the diagonal in $V \times V$, that is on vectors of the form $(v, v) \in V \times V$.

One can use Sylvester's law of inertia to prove the following fact from linear algebra.

Proposition 2.3. Let $(V, \langle \cdot, \cdot \rangle)$ be a pseudo-Euclidean vector space of dimension n and let ν denote the index of $\langle \cdot, \cdot \rangle$. Then $(V, \langle \cdot, \cdot \rangle)$ is isometric to $(\mathbb{R}^n, \langle \cdot, \cdot \rangle_\nu)$, where

$$\langle v, v \rangle_\nu := \sum_{i=1}^{n-\nu} \left(v^i\right)^2 - \sum_{i=n-\nu+1}^{n} \left(v^i\right)^2.$$

Recall the definition of orthogonality from linear algebra:

Definition 2.4. Let $(V, \langle \cdot, \cdot \rangle)$ be a pseudo-Euclidean vector space and $W \subset V$ a pseudo-Euclidean linear subspace, meaning that $\langle \cdot, \cdot \rangle|_{W \times W}$ is a pseudo-Euclidean scalar product on W. Then the **orthogonal complement** $W^\perp \subset V$ of W in V with respect to $\langle \cdot, \cdot \rangle$ is given by

$$W^\perp := \{v \in V \mid \langle v, w \rangle = 0 \quad \forall w \in W\}.$$

$W^\perp$ is a linear subspace of V of dimension $\dim(W^\perp) = \dim(V) - \dim(W)$ and

$$W \oplus W^\perp = V.$$

If $W \subset V$ is any linear subspace of V, we will also use the notation $W^\perp$ for its orthogonal complement. Two arbitrary vectors $v, w \in V$ are called **orthogonal** if $\langle v, w \rangle = 0$, and two linear subspaces V_1, V_2 of V are called **orthogonal** to each other if $\langle v_1, v_2 \rangle = 0$ for all $v_1 \in V_1$, $v_2 \in V_2$. A basis $\{v_1, \ldots, v_n\}$ of V is called **orthogonal basis** with respect to $\langle \cdot, \cdot \rangle$ if $\langle v_i, v_j \rangle = 0$ for all $1 \leq i, j \leq n$, $i \neq j$. An orthogonal basis is called **orthonormal basis** if additionally $\|v_i\| = 1$ for all $1 \leq i \leq n$.

The following exercise recaptures some additional facts from linear algebra.

Exercise 2.5.

(i) Prove that every pseudo-Euclidean vector space admits an orthonormal basis.

[24] Compare this formula to the pullback of 1-forms.

(ii) Show that the index ν of a pseudo-Euclidean scalar product coincides with the number of elements in $\{i \mid \langle v_i, v_i\rangle = -1\}$ for any given orthonormal basis $\{v_1, \ldots, v_n\}$ of $(V, \langle\cdot,\cdot\rangle)$.

(iii) For any given pseudo-Euclidean vector space $(V, \langle\cdot,\cdot\rangle)$ and $W \subset V$ any linear subspace, prove:

   (a) $(W^{\perp})^{\perp} = W$,

   (b) W is a pseudo-Euclidean linear subspace $\Leftrightarrow W \cap W^{\perp} = \{0\} \Leftrightarrow V = W \oplus W^{\perp}$.

(iv) Linear isometries map orthonormal (orthogonal) bases to orthonormal (orthogonal) bases.

We want to translate the concept of pseudo-Euclidean vector spaces to smooth manifolds. More precisely we want to specify what it means to specify for each point p in a given manifold M a pseudo-Euclidean scalar product on T_pM, such that this assignment varies smoothly on M.

Definition 2.6. Let M be a smooth manifold. A **pseudo-Riemannian metric with index** $0 \leq \nu \leq \dim(M)$ on M is a symmetric $(0,2)$-tensor field $g \in \mathcal{T}^{0,2}(M)$, $g : p \mapsto g_p \in \mathrm{Sym}^2(T_p^*M)$, such that for all $p \in M$ g_p is a pseudo-Euclidean scalar product of index ν on T_pM. This in particular means that

$$g(X,Y) = g(Y,X) \in C^\infty(M)$$

for all vector fields $X, Y \in \mathfrak{X}(M)$. If $\nu = 0$, g is called **Riemannian metric**. In local coordinates $(x^1, \ldots, x^n)$ on $U \subset M$, g is of the form

$$g = \sum_{i,j=1}^{n} g_{ij} dx^i \otimes dx^j,$$

where

$$g_{ij} := g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \in C^\infty(U)$$

for all $1 \leq i,j \leq n$. The symmetry condition for g is equivalent to requiring that in all local coordinates $g_{ij} = g_{ji}$. This means that (g_{ij}), viewed as a $n \times n$-matrix valued smooth map on the coordinate domain, is at each point a symmetric matrix. If we write in local coordinates $X = \sum\limits_{i=1}^{n} X^i \frac{\partial}{\partial x^i}$, $Y = \sum\limits_{i=1}^{n} Y^i \frac{\partial}{\partial x^i}$, we obtain the local formula for $g(X,Y)$

$$g(X,Y) = \sum_{i,j=1}^{n} g_{ij} X^i Y^j,$$

which heuristically corresponds to plugging in X in the left half and Y in the right half of the tensor terms in g.

Definition 2.7. A smooth manifold M equipped with a (pseudo)-Riemannian metric g is called **(pseudo)-Riemannian manifold**.

Remark 2.8. Of particular importance in mathematics and physics are pseudo-Riemannian manifolds of index 0 and 1, that is **Riemannian manifolds** and **Lorentz manifolds**, respectively. The latter are the manifolds that are studied in general relativity, for an introduction see [O, Ch. 12]. Why would one want to study Riemannian manifolds in their full generality, aside from an explanation how the standard Riemannian metric on $\mathbb{R}^n$ induced by the Euclidean scalar product at each point transforms? The answer is manifold (this time, the latter is an adjective). First and foremost because it allows our studied geometrical objects to have **curvature**. We will study this topic extensively in Sections **??** and **??**. Furthermore, Riemannian metrics give us a way to study volumes of submanifolds. This is not completely trivial, as it involves the construction a of so-called **volume form** from a given Riemannian metric, respectively its restriction to submanifolds, cf. Section **??**. For starters, it allows us to define the arc-length of a curve.

Definition 2.9. Let (M, g) be a Riemannian manifold and $\gamma : I \to M$ a smooth curve. Then the **arc-length**, or simply **length**, of γ is defined as

$$L(\gamma) = \int_I \sqrt{g(\gamma', \gamma')}dt.$$

Note that $L(\gamma) = \infty$ is allowed.

Now we will study some explicit examples of pseudo-Riemannian manifolds.

Example 2.10.

(i) Any pseudo-Riemannian vector space $(V, \langle\cdot,\cdot\rangle)$ is, viewed as a smooth manifold with $g_p := \langle\cdot,\cdot\rangle$ for all $p \in V$[25]. If $V = \mathbb{R}^n$ equipped with its canonical coordinates and Euclidean scalar product at each tangent space, the induced Riemannian metric in canonical coordinates $(u^1, \dots, u^n)$ is given by

$$g = \sum_{i=1}^{n} du^i \otimes du^i.$$

(ii) Any smooth submanifold $M \subset \mathbb{R}^n$ equipped with

$$g \in \mathcal{T}^{0,2}(M), \quad g_p = \langle\cdot,\cdot\rangle|_{T_pM\times T_pM}$$

for all $p \in M$, that is the restriction of the Euclidean scalar product at origin $p \in \mathbb{R}^n$ to the tangent space of M at p.

(iii) More generally, any smooth submanifold of a smooth Riemannian manifold is by restriction of the metric to the tangent bundle of the smooth submanifold a Riemannian manifold.

(iv) If (M, g_M) and (N, g_N) are pseudo-Riemannian manifolds and g_M, g_N, have index ν_M, ν_N, respectively, the product $M \times N$ is a pseudo-Riemannian manifold of index $\nu_M + \nu_N$. The metric on $M \times N$ is given by

$$g_{M\times N} := g_M + g_N, \quad g_{M\times N}((v, w),(v, w)) = g_M(v, v) + g_N(w, w),$$

for all $(v, w) \in TM \oplus TN \cong T(M \times N)$. The metric $g_{M\times N}$ is called **product metric**.

Example 2.10 (iii) motivates the following definition.

Definition 2.11. Let $(N, \overline{g})$ be a pseudo-Riemannian manifold and $M \subset N$ a smooth submanifold. M is called **pseudo-Riemannian submanifold** of N if

$$g := \overline{g}|_{TM\times TM}$$

is a pseudo-Riemannian metric on M. In the above equation, the restriction to $TM \times TM$ means that we restrict the basepoint of $\overline{g}$ to $M \subset N$ and the vectors we are allowed to plug in to vectors in $TM \subset TN$.

Exercise 2.12.

(i) Show that any smooth manifold can be equipped with a Riemannian metric. [Hint: Use a countable smooth partition of unity subordinate to a countable atlas on M.]

[25] Recall that $T_pV \cong V$ for all $p \in V$.

(ii) Show that not every manifold can be equipped with a pseudo-Riemannian, not Riemannian, metric. This is to be understood to also exclude the index $\nu = \dim(M)$. ["Hint": This exercise is very difficult.]

(iii) Show that every $n \geq 2$-dimensional pseudo-Riemannian manifold N with metric $\overline{g}$ of index $1 \leq \nu \leq n-1$ has smooth submanifolds that are not pseudo-Riemannian submanifolds.

The pseudo-Riemannian manifold-analogue to isometries of pseudo-Euclidean vector spaces is as follows.

Definition 2.13. Let (M, g) and (N, h) be pseudo-Riemannian manifolds and $F : M \to N$ a diffeomorphism. Then F is called an **isometry** if $F^*h = g$ or, equivalently, $F_*g = h$. One checks that the first condition is equivalent to

$$g_p(X_p, Y_p) = h_{F(p)}(dF_p(X_p), dF_p(Y_p))$$

for all $X, Y \in \mathfrak{X}(M)$ and all $p \in M$, meaning that pointwise dF_p is a linear isometry. The two pseudo-Riemannian manifolds (M, g) and (N, h) are then called **isometric**. Note that the isometries $F : M \to M$ with respect to g form a group, **the isometry group of** (M, g), which is denoted by $\mathrm{Isom}(M, g)$.

Example 2.14.

(i) Every orthogonal transformation $A \in \mathrm{O}(n+1)$ is, by definition, an isometry of $\mathbb{R}^{n+1}$ equipped with the standard Riemannian metric given pointwise by the Euclidean scalar product $\langle \cdot, \cdot \rangle$.

(ii) Since each $A \in \mathrm{O}(n+1)$ restricts to a diffeomorphism of $S^n \subset \mathbb{R}^{n+1}$, it is an isometry of $(S^n, \langle \cdot, \cdot \rangle|_{TS^n \times TS^n})$. The Riemannian metric $\langle \cdot, \cdot \rangle|_{TS^n \times TS^n}$ is sometimes called the **round metric**.

(iii) Consider the upper half plane $H := \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ equipped with the Riemannian **Poincaré metric**

$$g = \frac{1}{y^2}(dx^2 + dy^2),$$

called the **Poincaré half-plane model**. When viewed as a subset of $\mathbb{C}$ via $H \ni (x, y) \mapsto x + iy \in \mathbb{C}$, one obtains an **isometric action**[26] of

$$\mathrm{PSL}(2, \mathbb{R}) = \mathrm{SL}(2, \mathbb{R})/_{\sim}, \quad A \sim B :\Leftrightarrow A = \pm B,$$

on $H \subset \mathbb{C}$ defined by

$$\mu : \mathrm{PSL}(2, \mathbb{R}) \times H \to H, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z := \frac{az + b}{cz + d}.$$

Exercise 2.15. Prove the statement in Example 2.14 (iii) and show that the group action $\mu : \mathrm{PSL}(2, \mathbb{R}) \times H \to H$ is transitive.

A change of coordinates on M induces a, pointwise, change of basis in TM. We obtain the following result for local forms of pseudo-Riemannian metrics under a change of coordinates.

[26]This means: A group action $\mu : \mathrm{PSL}(2, \mathbb{R}) \times H \to H$, where for every group element A fixed, the induced map $\mu(A, \cdot) : H \to H$ is an isometry.

Lemma 2.16. Let (M,g) be a pseudo-Riemannian manifold and $\varphi = (x^1,\ldots,x^n)$, $\psi = (y^1,\ldots,y^n)$, be local coordinate systems on $U \subset M$, respectively $V \subset M$, such that $U \cap V \neq \emptyset$. Denote on $U \cap V$

$$g = \sum_{i,j} g_{ij} dx^i \otimes dx^j = \sum_{i,j} \widetilde{g}_{ij} dy^i \otimes dy^j. \tag{2.1}$$

The local coordinate systems φ and ψ are related by $(x^1,\ldots,x^n) = F(y^1,\ldots,y^n)$ on $U \cap V$, where $F : \psi(U \cap V) \to \varphi(U \cap V)$. Then the matrix valued maps (g_{ij}) and $(\widetilde{g}_{ij})$ in (2.1) are related by

$$(\widetilde{g}_{ij})|_p = dF^T_{\psi(p)} \cdot (g_{ij})|_{\varphi^{-1}(F(\psi(p)))} \cdot dF_{\psi(p)}.$$

Proof. Follows by considering coordinate representations of (g_{ij}) and $(\widetilde{g}_{ij})$, writing down the pullback of (g_{ij}) with respect to F, and comparing the prefactors. □

The above lemma might look more complicated than it is at first glance. Pointwise, the statement is precisely the transformation law for pseudo-Euclidean scalar products under a change of basis.

Recall the following construction from linear algebra.

Definition 2.17. Let V be a real finite-dimensional vector space and $A \in \mathrm{End}(V) \cong V \otimes V^*$, so that for a basis $\{v_1,\ldots,v_n\}$ of V

$$A = \sum_{i,j=1}^n a^i{}_j v_i \otimes v_j^*.$$

The **trace** of A is defined as

$$\mathrm{tr}(A) := \sum_{i=1}^n a^i{}_i.$$

Exercise 2.18. Check that the definition of the trace in Definition 2.17 is well-defined, meaning that it gives the same value for all choices of a basis of V.

Example 2.19.

(i) $\mathrm{tr}(\mathrm{id}_V) = \dim(V)$,

(ii) $\mathrm{tr}(A+B) = \mathrm{tr}(A) + \mathrm{tr}(B)$ for all $A, B \in \mathrm{End}(V)$,

(iii) $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for all $A, B \in \mathrm{End}(V)$,

(iv) $\mathrm{tr}(v \otimes \omega) = \omega(v)$ for all $v \in V$, $\omega \in V^*$.

One can furthermore show the following:

Lemma 2.20. Let $(V, \langle\cdot,\cdot\rangle)$ be a finite-dimensional pseudo-Euclidean vector space and $A \in \mathrm{End}(V)$. Let $\{e_1,\ldots,e_n\}$ be an orthonormal basis of V with respect to $\langle\cdot,\cdot\rangle$. Then

$$\mathrm{tr}(A) = \sum_{i=1}^n \varepsilon_i \langle e_i, Ae_i\rangle,$$

where $\varepsilon_i := \langle e_i, e_i\rangle \in \{-1, 1\}$ for all $1 \leq i \leq n$.

Proof. Exercise. □

One can for pseudo-Euclidean vector spaces further define natural (possibly indefinite) scalar product on $V^{\otimes r} \otimes (V^*)^{\otimes s}$ for all $r + s > 0$.

Definition 2.21. Let $(V, \langle\cdot,\cdot\rangle)$ be a pseudo-Euclidean vector space and $\{e_1, \ldots, e_n\}$ a basis of V. Let further $A \in V^{\otimes r} \otimes (V^*)^{\otimes s}$ and write

$$\langle\cdot,\cdot\rangle = \sum_{i,j=1}^{n} g_{ij} e_i^* \otimes e_j^*,$$
$$A = \sum_{\substack{1 \le i_1,\ldots,i_r \le n \\ 1 \le j_1,\ldots,j_r \le n}} A^{i_1\ldots i_r}{}_{j_1\ldots j_s} e_{i_1} \otimes \ldots \otimes e_{i_r} \otimes e_{j_1}^* \otimes \ldots \otimes e_{j_s}^*.$$

Then

$$\langle A, A\rangle := \sum_{\substack{1 \le i_1,\ldots,i_r \le n \\ 1 \le j_1,\ldots,j_r \le n \\ 1 \le I_1,\ldots,I_r \le n \\ 1 \le J_1,\ldots,J_r \le n}} A^{i_1\ldots i_r}{}_{j_1\ldots j_s} \cdot A^{I_1\ldots I_r}{}_{J_1\ldots J_s} \cdot g_{i_1 I_1} \cdot \ldots \cdot g_{i_r I_r} \cdot g^{j_1 J_1} \cdot \ldots \cdot g^{j_s J_s} \tag{2.2}$$

defines a, possibly indefinite, symmetric bilinear form on $V^{\otimes r} \otimes (V^*)^{\otimes s}$. In the above formula the g-terms fulfil, when viewed as a symmetric matrix,

$$(g^{ij}) := (g_{ij})^{-1}.$$

Remark 2.22. Formula (2.2) should make you ask one thing and realize another. Firstly you should ask why one would write down something like that. It actually, when generalized to smooth manifolds and tensor powers of the tangent bundle, is used to define certain geometric invariants, e.g. the so-called **Kretschmann scalar**. Secondly, the summation ranges in (2.2) should convince you that it might sometimes be a good idea to be a little bit imprecise to increase readability when the ranges are clear from the setting. In the following we will do just that.

As hinted in the above remark, Definitions 2.17 and 2.21 readily generalize to smooth manifolds and tensor bundles.

Definition 2.23. Let (M, g) be a pseudo-Riemannian manifold, $A \in \mathcal{T}^{1,1}(M)$ an endomorphism field, $h \in \mathcal{T}^{0,2}(M)$ a symmetric $(0,2)$-tensor field, and $B \in \mathcal{T}^{r,s}(M)$ for $r + s > 0$ an arbitrary tensor field. Then the **trace of** A is in local coordinates $(x^1, \ldots, x^n)$, so that $A = \sum A^i{}_j \frac{\partial}{\partial x^i} \otimes dx^j$, given by

$$\mathrm{tr}(A) := \sum_i A^i{}_i.$$

The above term is **invariant under coordinate change**, which follows from fibrewise invariance of the choice of basis in T_pM and the fact that the coordinate cotangent vector at each point are precisely the dual to the coordinate tangent vectors at that point. This means that $\mathrm{tr}_g(A) \in C^\infty(M)$. The trace of h with respect to g is defined in local coordinates as

$$\mathrm{tr}_g(h) := \sum_{i,j} h_{ij} g^{ij}.$$

As for the endomorphism field, $\mathrm{tr}_g(h)$ is **invariant under coordinate change**, but **not** invariant of the pseudo-Riemannian metric g. Furthermore, we define the induced pairing of B with itself with respect to g in the given local coordinates as

$$g(B, B) := \sum B^{i_1\ldots i_r}{}_{j_1\ldots j_s} \cdot B^{I_1\ldots I_r}{}_{J_1\ldots J_s} \cdot g_{i_1 I_1} \cdot \ldots \cdot g_{i_r I_r} \cdot g^{j_1 J_1} \cdot \ldots \cdot g^{j_s J_s},$$

where

$$B = \sum B^{i_1\ldots i_r}{}_{j_1\ldots j_s} \frac{\partial}{\partial x^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_s}$$

and $(g^{ij}) = (g_{ij})^{-1}$ at each point when viewed as a symmetric matrix valued map. As for the trace, value of $g(B,B)$ does not depend on the choice of local coordinates which implies $g(B,B) \in C^\infty(M)$. Note that one can similarly define a symmetric pairing g in the bundle $T^{r,s}M \to M$, which is an example of a possibly indefinite **bundle metric**.

Example 2.24. For any pseudo-Riemannian manifold (M,g) of dimension n we have

$$\mathrm{tr}_g(g) = g(g,g) \equiv n.$$

Pseudo-Riemannian metrics allow us to take any (r,s)-tensor field and change it into an (r',s')-tensor field if $r+s = r'+s'$. This process is reversible, and on the level of bundles known as **musical isomorphisms**.

Proposition 2.25. Let (M,g) be a pseudo-Riemannian manifold. Then $T^{r,s}M \to M$ and $T^{r',s'}M \to M$ are isomorphic as vector bundles if $r+s = r'+s'$.

Proof. We first proof that $T^*M \to M$ and $TM \to M$ are isomorphic. Let

$$F : TM \to T^*M, \quad v \mapsto g(v,\cdot).$$

It is clear that $g(v,\cdot) \in T_p^*M$ for all $v \in T_pM$. Furthermore, the map F is smooth, fibre-preserving, and at each point a linear isomorphism. Its inverse is given by

$$F^{-1} : T^*M \to TM, \quad \omega \mapsto g^{-1}(\omega,\cdot),$$

where we use the pointwise identification $(T_p^*M)^* = T_pM$ and g^{-1} is given in local coordinates by

$$g^{-1} = \sum g^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}.$$

In order to show that $T^{r,s}M \to M$ and $T^{r',s'}M \to M$ are isomorphic for arbitrary r,s,r',s' with $r+s = r'+s'$ one inductively uses entrywise isomorphisms. Note that there are usually choices involved which vector or covector parts to change into covector and vector parts, respectively. These choices correspond to which index is lowered or raised. Exceptions are e.g. going from $T^{1,1}M$ to $T^{0,2}M$. Care is also required when composing such isomorphisms as it might lead to "swapping" in the tensor powers of the vectors and covectors, where we recall that e.g. in $TM \otimes TM$ swapping the fibres is an isomorphism of vector bundles. □

The above Proposition 2.25 describes what is also known as lowering/raising indices. This is due to when one composes these isomorphisms with tensor fields, locally the prefactors' index locations change from up to down or the other way round. Check for example what happens to the used pseudo-Riemannian metric if one raises an index!

Remark 2.26. The isomorphisms of vector bundles $T^{r,s}M \to T^{r+1,s-1}M$ are denoted by $\sharp$ (read: "sharp"), and the isomorphisms $T^{r,s}M \to T^{r-1,s+1}M$ are denoted by $\flat$ (read: "flat"). Hence the name "musical isomorphisms". One needs to make sure to be aware of **which index** is moved up or down if there is a choice! Note that, using the musical isomorphisms, we could have defined the trace of endomorphism fields $A \in \mathfrak{T}^{1,1}(M)$ on a pseudo-Riemannian manifold (M,g) as

$$\mathrm{tr}_g(A) = \sum_{ij} (\sharp A)^{ij} g_{ij}.$$

It is crucial to observe that, as for our definition of $\mathrm{tr}(A)$ in Definition 2.23, the above term $\mathrm{tr}_g(A)$ is invariant of the pseudo-Riemannian metric g[27]. Also note that commonly one suppresses

[27] Ask yourself why this is true!

writing down $\sharp$ and $\flat$ and simply writes e.g. A^{ij} instead of $(\sharp A)^{ij}$ since the location of the indices (i.e. "up" or "down") determine which of them has been raised or lowered. It is however of **prime importance** to always be aware of **which metric** has been used to lower or raise indices!

Recall the gradient of smooth functions on $\mathbb{R}^n$. The gradient of $f : \mathbb{R}^n \to \mathbb{R}$ is a vector field given by

$$\operatorname{grad}(f) := \sum_{i=1}^{n} \frac{\partial f}{\partial u^i} \frac{\partial}{\partial u^i} \in \mathfrak{X}(\mathbb{R}^n).$$

There is an invariant generalization for that concept to pseudo-Riemannian manifolds using our above defined musical isomorphisms for which the above formula is precisely the to-be-defined gradient of f on $\mathbb{R}^n$ with respect to the Riemannian metric given by the standard Euclidean scalar product (in **each** tangent space $T_p\mathbb{R}^n$).

Definition 2.27. Let (M, g) be a pseudo-Riemannian manifold and $f \in C^\infty(M)$ a smooth function. The **gradient vector field of f with respect to** g, $\operatorname{grad}_g(f) \in \mathfrak{X}(M)$, is defined as

$$\operatorname{grad}_g(f) := g^{-1}(df) \in \mathfrak{X}(M).$$

In local coordinates $(x^1, \ldots, x^n)$, $\operatorname{grad}_g(f)$ is of the form

$$\sum_{i,j=1}^{n} \frac{\partial f}{\partial x^i} g^{ij} \frac{\partial}{\partial x^j}.$$

Gradient vector fields are of critical importance in the study of pseudo-Riemannian submanifolds as we find the following description of tangent bundle of pseudo-Riemannian submanifolds.

Lemma 2.28. Let $(\overline{M}, g)$ be a pseudo-Riemannian manifold, $M \subset \overline{M}$ a pseudo-Riemannian submanifold of codimension k, and identify $T_qM = \iota_*(T_qM) \subset T_q\overline{M}$ for all $q \in M$, where ι is the inclusion. For $p \in M$ fixed let[28] $f = (f^1, \ldots, f^k) : U \to \mathbb{R}^k$, $U \subset \overline{M}$ open, $p \in U$, be any smooth map of maximal rank such that

$$M \cap U = \{f = 0\} \subset \overline{M}.$$

Then

$$T_qM = \ker(df_q^1) \cap \ldots \cap \ker(df_q^k) \subset T_q\overline{M} \tag{2.3}$$

and

$$(T_qM)^\perp = \operatorname{span}_{\mathbb{R}}\{\operatorname{grad}_g(f^1)_q, \ldots, \operatorname{grad}_g(f^k)_q\} \subset T_q\overline{M} \tag{2.4}$$

for all $q \in M \cap U$. In particular, $T_qM \oplus (T_qM)^\perp = T_q\overline{M}$ for all $q \in M \cap U$.

Proof. Fix $q \in M \cap U$ and $v \in T_qM$. For any smooth curve $\gamma : I \to M \subset \overline{M}$, $\gamma'(t)$ is tangential to M for all $t \in I$, which follows by using adapted coordinates. Choose a smooth curve $\gamma : (-\varepsilon, \varepsilon) \to M \subset \overline{M}$ fulfilling $\gamma'(0) = v$. Then for all $1 \le i \le k$,

$$df^i(v) = \left.\frac{\partial}{\partial t}\right|_{t=0} (f \circ \gamma) = df^i(v) = \left.\frac{\partial}{\partial t}\right|_{t=0} (0) = 0.$$

This shows $T_qM \subset \ker(df_q^1) \cap \ldots \cap \ker(df_q^k)$. On the other hand, f being of maximal rank implies that $df_q^1, \ldots, df_q^k$ are linearly independent. Hence, the intersection of their kernels fulfils

$$\dim(\ker(df_q^1) \cap \ldots \cap \ker(df_q^k)) = \dim(T_q\overline{M}) - k = \dim(T_qM).$$

Hence, (2.3) holds as claimed. For (2.4) one uses that g is pointwise nondegenerate, hence each nonzero vector in $\operatorname{span}_{\mathbb{R}}\{\operatorname{grad}_g(f^1)_q, \ldots, \operatorname{grad}_g(f^k)_q\}$ is not contained in $T_qM = \ker(df_q^1) \cap \ldots \cap \ker(df_q^k)$. By $T_qM \oplus (T_qM)^\perp = T_q\overline{M}$ and comparing dimensions, (2.4) follows. □

[28] If you are not convinced of the existence of such a function f near any point: Prove its existence!

The above lemma tells us how to pointwise understand the tangent space of an ambient manifold of a submanifold as a combination of **tangent and normal parts**. How can we formulate this in a coordinate free, global statement? To do so we need to define bundles along submanifolds.

Lemma 2.29. Let $\pi_E : E \to \overline{M}$ be a vector bundle of rank k and M be a submanifold of $\overline{M}$. Then

$$\pi_{E|_M} : E|_M \to M, \quad (E|_M)_p := \pi_{E|_M}^{-1}(p) := \pi_E^{-1}(p) \quad \forall p \in M, \quad E|_M := \bigsqcup_{p \in M} (E|_M)_p,$$

is a vector bundle of rank k over M. It is called **vector bundle along** M.

Proof. In order to proof this statement it suffices to work in local coordinates. Without loss of generality assume that locally, M is given by an open set in $\mathbb{R}^\ell$, $\ell \leq \dim(\overline{M})$, and the inclusion $M \subset \overline{M}$ is of the form

$$\iota : (x^1, \dots, x^\ell) \mapsto (x^1, \dots, x^\ell, 0, \dots, 0) \in \mathbb{R}^{\dim(\overline{M})}.$$

The rest of the proof consists of applying the vector bundle chart lemma to the restriction of, after possibly shrinking U, the transition functions of $E \to \overline{M}$ in local coordinates to $U \subset \mathbb{R}^{\dim(\overline{M})}$ and observing that the vector parts are, still, smooth. □

The above lemma might seem more complicated than it actually is. It means that locally, we make the base space smaller in dimension but keep all possible vectors attached to that smaller set. The most important example for us is restricting the tangent and cotangent bundle of an ambient manifold to a submanifold. In this setting, we see that vector fields along the inclusion map are just sections of the tangent bundle of the ambient manifold along the submanifold. Lemmas 2.28 and 2.29 motivate the following definition.

Definition 2.30. Let $(\overline{M}, g)$ be a pseudo-Riemannian manifold and $M \subset \overline{M}$ a pseudo-Riemannian submanifold of codimension k. Then the **normal bundle of** M, $TM^\perp \to M$, is defined as

$$TM^\perp := \bigsqcup_{p \in M} (T_pM)^\perp,$$

with projection induced by the tangent bundle of $\overline{M}$ along M, $T\overline{M}|_M \to M$. In particular we have

$$T\overline{M}|_M = TM \oplus TM^\perp,$$

and the above direct sum is orthogonal with respect to g.

In Definition 2.30 above we have split up $T\overline{M}|_M$ into $TM \oplus TM^\perp$, so in particular we have at each point $p \in M$

$$T_p\overline{M}|_M = T_p\overline{M} = T_pM \oplus T_pM^\perp.$$

This means that pointwise, we have that e.g. T_pM is a subvector space of $T_p\overline{M}$. Is this a special case of a more general concept for bundles? As you might have guessed already, the answer is yes.

Definition 2.31. Let $\pi_E : E \to M$ be a vector bundle. Another vector bundle $\pi_F : F \to M$ is called **subbundle of** $E \to M$ if for all $p \in M$, F_p is a linear subspace of E_p, the canonical injection

$$F \hookrightarrow E,$$

given fibrewise by the inclusion $F_p \subset E_p$, is an embedding, $\pi_F = \pi_E|_F$, and for all local trivializations ϕ of E the restrictions $\phi|_F$ are local trivializations of F. This means that the bundle structure of $F \to M$ and the smooth manifold structure of the total space F are induced by the bundle structure of $E \to M$ and the smooth manifold structure of the total space E, respectively.

In the sense of Definition 2.31 TM and $TM^\perp$ of a pseudo-Riemannian submanifold $M \subset \overline{M}$ are both subbundles of $T\overline{M}|_M$.

Exercise 2.32. Give a rigorous proof of the above statement. [Hint: You will probably learn exactly the same and at the same time gain more geometrical insight if you prove this statement for surfaces in $\mathbb{R}^3$, while at the same time not having to fight with too many indices.]

The most prominent examples of gradient vector fields and their relation to the normal bundle that are usually used for introductory purposes are level sets of quadratic polynomials which fulfil a certain nondegeneracy condition.

Example 2.33.

(i) Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$, $f(u^1, \dots, u^n) = \sum\limits_i (u^i)^2$ and consider the ambient space $\mathbb{R}^{n+1}$ equipped with its standard Riemannian metric, denoted simply by $\langle \cdot, \cdot \rangle$. Then

$$S^n = \{f = 1\} \subset \mathbb{R}^{n+1}$$

is a Riemannian submanifold of $(\mathbb{R}^{n+1}, \langle \cdot, \cdot \rangle)$ with induced Riemannian metric

$$g := \langle \cdot, \cdot \rangle|_{TS^n \times TS^n}.$$

The normal bundle of S^n realized as a submanifold of $(\mathbb{R}^{n+1}, \langle \cdot, \cdot \rangle$, $TS^{n\perp}$, is spanned by the **position vector field** $\xi \in \mathfrak{X}(\mathbb{R}^{n+1})$ along S^n,

$$\xi : p \mapsto p \quad \forall p \in \mathbb{R}^{n+1},$$

where we have as usual identified $T_p\mathbb{R}^{n+1}$ with $\mathbb{R}^{n+1}$ for all $p \in \mathbb{R}^{n+1}$. The tangent bundle of TS^n, viewed as a subbundle of $T\mathbb{R}^{n+1}|_{S^n}$, is thus fibrewise given by

$$T_pS^n = \ker(\langle \xi_p, \cdot \rangle) \subset T_p\mathbb{R}^{n+1}.$$

This means that a vector field X along S^n is tangential to S^n if and only if $\langle \xi, X \rangle \equiv 0$. Note that the function f used to define S^n fulfils $f = \langle \xi, \xi \rangle$.

(ii) Next consider $\mathbb{R}^{n+1}$ but now equipped with a pseudo-Riemannian metric given in canonical coordinates by

$$\langle \cdot, \cdot \rangle_\nu := \sum_{i=1}^{n-\nu} du^i \otimes du^i - \sum_{i=n-\nu+1}^{n} du^i \otimes du^i.$$

Let $\xi \in \mathfrak{X}(\mathbb{R}^{n+1})$ denote the position vector field and define $f : \mathbb{R}^{n+1} \to \mathbb{R}$, $f := \langle \xi, \xi \rangle$. Then the level sets $\{f = -1\}$[29] are called **hyperboloids**,

$$H^n_\nu := \left\{ \langle \xi, \xi \rangle = \sum_{i=1}^{n-\nu+1} \left(u^i\right)^2 - \sum_{i=n-\nu+2}^{n+1} \left(u^i\right)^2 = -1 \right\} \subset \mathbb{R}^{n+1}.$$

Hyperboloids in $(\mathbb{R}^{n+1}, \langle \cdot, \cdot \rangle_\nu)$ are n-dimensional pseudo-Riemannian manifolds with induced pseudo-Riemannian metric of index $\nu - 1$. As for S^n,

$$T_pH^n_\nu = \ker(\langle \xi_p, \cdot \rangle_\nu) \subset T_p\mathbb{R}^{n+1}$$

[29] Be aware of the sign!

and

$$T_pH^{n\perp}_\nu = \mathbb{R}\xi_p,$$

where $\mathbb{R}\xi_p$ is another commonly used notation for the linear span of **one** vector, that is $\mathrm{span}_\mathbb{R}\{\xi_p\}$. In the case $n = 3$, $\nu = 1$, H^3_1 is known as **two-sheeted hyperboloid**, and for $n = 3$, $\nu = 2$, H^3_2 is the **one-sheeted hyperboloid**.

Exercise 2.34. Prove the claims in Example 2.33.

In Example 2.33 we used the term that a vector field spans a vector bundle. Conceptually, this belongs in the setting of frames of vector bundles, which generalize the concept of a basis of a vector space.

Definition 2.35. Let $E \to M$ be a vector bundle of rank k. Then a **(local) frame of** E over $U \subset M$, U open, is a set of k (local) sections

$$\{s_i \in \Gamma(E|_U),\ 1 \le i \le k\},$$

such that for all $p \in U$ fixed, the vectors $s_i(p) \in E_p$, $1 \le i \le k$, are linearly independent. Equivalently,

$$\mathrm{span}_\mathbb{R}\{s_i(p) \in E_p \mid 1 \le i \le k\} = E_p$$

for all $p \in U$.

Exercise 2.36. Show that every local section $s \in \Gamma(E|_U)$ in a vector bundle E can be written as a $C^\infty(U)$-linear combination of the elements of a local frame of $E \to M$ over U. Check that these prefactors in $C^\infty(U)$ are uniquely determined for any given local section.

Local frames are very useful in order to check if subsets of a certain form of given vector bundles are subbundles.

Lemma 2.37. Let $E \to M$ be a vector bundle of rank k and suppose that for $\ell \le k$ we are given a linear subspace $F_p \subset E_p$ of constant dimension ℓ for all $p \in M$. Then $\bigsqcup_{p\in M} F_p \to M$ is, with all data necessary induced by $E \to M$, a subbundle of $E \to M$ if and only if for every $p \in M$ we can find a local frame $\{s_1, \dots, s_k\}$ of $E|_U \to U$, $U \subset M$ an open neighbourhood of p, such that for all $q \in U$, $\{s_1(q), \dots, s_\ell(q)\}$ is a basis of F_q.

Proof. [L1, Lem. 10.32] □

Subbundles might look very complicated at first glance, but at least locally we can use the above lemma to always describe them as follows.

Lemma 2.38. Let $F \to M$ be a subbundle of rank ℓ of a vector bundles $E \to M$ of rank $k > \ell$. For any $p \in M$ we can find an open neighbourhood $U \subset M$ of p and a local trivialization of $E \to M$ over U, $\phi : E|_U \to U \times \mathbb{R}^k$, such that

$$\phi(\iota(F|_U)) = U \times \{(v^1, \dots, v^\ell, 0, \dots, 0) \mid (v^1, \dots, v^\ell) \in \mathbb{R}^\ell\} \subset U \times \mathbb{R}^k.$$

In the above equation, $\iota : F \hookrightarrow E$ denotes the inclusion map.

Proof. We use Lemma 2.37. Choose a local frame $\{s_1, \dots, s_k\}$ of $E \to M$ over $U \subset M$, such that $\{s_1, \dots, s_\ell\}$ is a local frame of $F \to M$ over U. The inverse of the smooth map

$$\eta : U \times \mathbb{R}^k \to E|_U, \quad (p, v^1, \dots, v^k) \mapsto \sum_{i=1}^k v^i s_i(p)$$

is smooth and a local trivialization of $E \to M$ over U. This follows from the implicit function theorem. We obtain

$$\eta^{-1}(\iota(F|_U)) = U \times \{(v^1, \ldots, v^\ell, 0, \ldots, 0) \mid (v^1, \ldots, v^\ell) \in \mathbb{R}^\ell\},$$

so setting $\phi = \eta^{-1}$ finishes the proof. □

Lemma 2.38 means that locally up to vector bundle isomorphisms, subbundles of vector bundles look like the inclusion in the first ℓ factors of the vector parts in $U \times \mathbb{R}^\ell \to U$ into $U \times \mathbb{R}^\ell \to U$.

In the special case of the tangent bundle of an n-dimensional smooth manifold, a local frame of $TM \to M$ over $U \subset M$ open is a set of n vector fields

$$\{X_i,\ 1 \leq i \leq n\}, \quad X_i \in \mathfrak{X}(U) \quad \forall 1 \leq i \leq n$$

such that for all $p \in U$, $\{(X_i)_p,\ 1 \leq i \leq n\}$ is a set of linearly independent vectors. This in particular means that

$$\mathrm{span}_{\mathbb{R}}\{(X_i)_p,\ 1 \leq i \leq n\} = T_pU \quad \forall p \in U,$$

and by Exercise 2.36 we can for each local vector field $X \in \mathfrak{X}(U)$ find a uniquely determined set of local functions $f_i \in C^\infty(U)$, $1 \leq i \leq n$ such that $X = \sum_{i=1}^{n} f_i X_i$.

Next we will use the language of local frames and subbundles to split up the $(0,2)$-tensor bundle $T^{0,2}M \to M$ over a smooth manifold into symmetric and antisymmetric parts. This construction and similar constructions are important to properly understand our upcoming study of curvature and the exterior differential on k-form analogues for smooth manifolds. Recall the following fact from linear algebra.

Lemma 2.39. Let V be a finite-dimensional real vector space with basis $\{v_1, \ldots, v_n\}$. Then

$$V \otimes V \cong \mathrm{Sym}^2(V) \oplus \Lambda^2 V,$$

where $\mathrm{Sym}^2(V) := \mathrm{span}_{\mathbb{R}}\{v_i \otimes v_j + v_j \otimes v_i,\ 1 \leq i, j \leq n\}$ and $\Lambda^2 V := \mathrm{span}_{\mathbb{R}}\{v_i \otimes v_j - v_j \otimes v_i,\ 1 \leq i, j \leq n\}$.

When viewed as matrices, the direct sum in Lemma 2.39 corresponds to writing a square matrix as its symmetric and antisymmetric parts, which are uniquely determined. One writes

$$v_i v_j := \frac{1}{2}(v_i \otimes v_j + v_j \otimes v_i), \quad v_i \wedge v_j := v_i \otimes v_j - v_j \otimes v_i,$$

and has $v_i \otimes v_j = v_i v_j + \frac{1}{2} v_i \wedge v_j$ for all $1 \leq i, j \leq n$. In particular $v_i v_i = v_i \otimes v_i$. This concept translates to local frames of vector bundles. We obtain the following.

Definition 2.40. Let M be a smooth manifold and $(x^1, \ldots, x^n)$ be local cooordinates on $U \subset M$. The **bundle of symmetric $(0,2)$-tensors** on M is the subbundle

$$\mathrm{Sym}^2(T^*M) \subset T^{0,2}M$$

with local frame over U given by $\left\{dx^i dx^j = \frac{1}{2}(dx^i \otimes dx^j + dx^j \otimes dx^i),\ 1 \leq i, j \leq n\right\}$. Sections of $\mathrm{Sym}^2(T^*M)$ are precisely symmetric $(0,2)$-tensor fields, which in particular includes all possible pseudo-Riemannian metrics on M. On the other hand we have the **anti-symmetric $(0,2)$-tensors** on M,

$$\Lambda^2 T^*M \subset T^{0,2}M,$$

which have local frame $\{dx^i \wedge dx^j = dx^i \otimes dx^j - dx^j \otimes dx^i,\ 1 \leq i, j \leq n\}$. Sections in $\Lambda^2 T^*M \to M$ are called **2-forms** and are denoted by $\Omega^2(M)$. Local sections in $\Lambda^2 T^*M \to M$ over $U \subset M$, U open, are denoted by $\Omega^2(U)$.

With Definition 2.40, a pseudo-Riemannian metric g on M can be written locally as

$$g = \sum_{i,j} g_{ij} dx^i \otimes dx^j = \sum_{i,j} g_{ij} dx^i dx^j.$$

Make sure you understand why the second equivalence in the above equation holds true! Also be aware that the rightmost sum in the above equation has a certain error potential when going from tensor to matrix notation. For example, the pseudo-Riemannian metric $dxdy$ on $\mathbb{R}^2$ with canonical coordinates (x, y) is in matrix notation given by

$$dx\, dy \text{ ``} = \text{''} \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}.$$

Make absolutely sure to understand this.

Now suppose that you are given a smooth manifold M and a symmetric $(0, 2)$-tensor field $g \in \mathcal{T}^{0,2}(M)$. At each point $p \in M$, you can associate to g_p a natural number called its index as follows.

Definition 2.41. The **index** of a symmetric $(0, 2)$-tensor field $g \in \mathcal{T}^{0,2}(M)$ at $p \in M$ is defined as

$$\nu(p) := \text{ number of negative eigenvalues of } g_p,$$

where g_p is viewed as symmetric matrix in local coordinates, i.e.

$$g_p = \sum_{ij} g_{ij}(p) dx^i \otimes dx^j.$$

Exercise 2.42. Show that the index in Definition 2.41 is actually well defined, that is does not depend on the choice of local coordinates.

How can we use the definition of the pointwise index of a symmetric $(0, 2)$-tensor field to test if it is a pseudo-Riemannian metric? The answer is as follows.

Proposition 2.43. Let M be a connected smooth manifold and $g \in \mathcal{T}^{0,2}(M)$ a symmetric $(0, 2)$-tensor field that is nondegenerate[30] at all points $p \in M$. Then g is a pseudo-Riemannian metric.

Proof. It suffices to show that the index $\nu : M \to \mathbb{N}_0$ is continuous, where $\mathbb{N}_0$ is equipped with the discrete topology. This follows from finish this!!! □

Next suppose that we are given just a smooth manifold and want to construct a pseudo-Riemannian metric. While this problem is usually difficult if the index of our metric is supposed to be positive (and not equal to the dimension of our manifold), for the Riemannian case, that is for vanishing index, we have the following nice result.

Proposition 2.44. Let M be a smooth manifold. Then there exists a Riemannian metric g on M.

Proof. This is Exercise 2.12 (i). We remark here that if g is a Riemannian metric and h is a symmetric $(0, 2)$-tensor with compact support, then for $\varepsilon > 0$ small enough $g + h$ will still be a Riemannian metric. This means that our constructed metric is far from unique. □

Remark 2.45. Riemannian metrics induce pointwise norm (for later with geodesic balls), points to relation to metric topology

[30] "nondegenerate" = at given point nondegenerate symmetric bilinear form

Example 2.46. counterexample to existence of Lorentz metric later! needs euler char, check bookmark

Definition 2.47. k-forms and corresponding bundle, exterior algebra structure, d-complex, cartans magic formula as exercise, induced volume form

Let us return to isometries of pseudo-Riemannian manifolds, cf. Definition 2.13. We already noted that they form a group, so it is reasonable to ask the following. It is clear that the identity is always an isometry, so how can we perturb it infinitesimally while preserving the isometry property? To answer that question we will use our knowledge of the Lie derivative and local flows of vector fields.

Proposition 2.48. *Let (M, g) be a pseudo-Riemannian manifold and let $X \in \mathfrak{X}(M)$. Suppose that for every local flow $\varphi : I \times U \to M$ of X, $\varphi_t \in \mathrm{Isom}(M, g)$ for all $t \in I$. Then $\mathcal{L}_X g = 0$. The converse statement also holds true.*

Proof. asdf ☐

Definition 2.49. Vector fields as in Proposition 2.48, that is $\mathcal{L}_X g = 0$ for $X \in \mathfrak{X}(M)$, (M, g) pseudo-Riemannian manifolds, are called **Killing[31] vector fields**.

Proposition 2.48 in particular means that Killing vector fields generate **local one parameter groups of isometries**. Killing vector fields, as a linear subspace of all vector fields, have the following structure algebraic property.

Lemma 2.50. *Let (M, g) be a pseudo-Riemannian manifold. Killing vector fields form a Lie subalgebra of $(\mathfrak{X}(M), [\cdot, \cdot])$, meaning that for any Killing vector fields $X, Y \in \mathfrak{X}(M)$, $[X, Y]$ is also a Killing vector field.*

Proof. Exercise. [Hint: Use the Jacobi identity $\mathcal{L}_{[X,Y]} Z = \mathcal{L}_X(\mathcal{L}_Y Z) - \mathcal{L}_Y(\mathcal{L}_X Z)$ for all $X, Y, Z \in \mathfrak{X}(M)$.] ☐

In fact, one can show more if M is Riemannian, but the proof of the following is beyond the scope of this course.

Theorem 2.51. *Let (M, g) be a Riemannian manifold of dimension n. The the Lie algebra of Killing vector fields is finite dimensional of dimension at most $\frac{1}{2}n(n + 1)$.*

Proof. [KN, Thm 3.3], in the corresponding chapter the structure of the isometry group as a Lie group acting on M is also treated. ☐

Let us look at some explicit examples of Killing vector fields.

Example 2.52.

(i) Let A be an $(n + 1) \times (n + 1)$ skew real matrix, that is $A^T = -A$. Then $e^{At} \in \mathrm{O}(n + 1)$ for all $t \in \mathbb{R}$. Then the vector field $X \in \mathfrak{X}(S^n)$ given by

$$X_p = \left.\frac{\partial}{\partial t}\right|_{t=0} \left(e^{At} p\right) \in T_p S^n$$

is a Killing vector field of the standard round metric on S^n, that is the restriction of the pointwise Euclidean scalar product in the ambient manifold $\mathbb{R}^{n+1}$. Note that $e^{A\cdot} : \mathbb{R} \times S^n \to \mathbb{R}^{n+1}$, $(t, v) \mapsto e^{At} v$ is the global flow of X.

[31] Wilhelm Karl Joseph Killing (1847 – 1923)

(ii) Consider $(\mathbb{R}^n, \langle\cdot,\cdot\rangle_\nu)$ for any $0 \leq \nu \leq n$ as in Example 2.33 and fix $(c^1,\ldots,c^n) \in \mathbb{R}^n$. Then $X \in \mathfrak{X}(\mathbb{R}^n)$, $X = \sum\limits_i c^i \frac{\partial}{\partial u^i}$, is a Killing vector field.

(iii) Let (M,g) and (N,h) be pseudo-Riemannian manifolds, X a Killing vector field on (M,g), and Y a Killing vector field on (N,h). Then $X+Y$ is a Killing vector field on $(M\times N, g\oplus h)$.

Now suppose that we are given a pseudo-Riemannian manifold and and do not know which vector fields are Killing vector fields. How do we approach this problem, at least locally?

Lemma 2.53. Let (M,g) be a pseudo-Riemannian manifold. Then $X \in \mathfrak{X}(M)$ is a Killing vector field if and and only if it fulfils

$$\sum_{k=1}^{n}\left(X^k\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial X^k}{\partial x^i}g_{jk} + \frac{\partial X^k}{\partial x^j}g_{ik}\right) = 0 \quad \forall 1 \leq i,j \leq n$$

for all local coordinates $(x^1,\ldots,x^n)$ on M.

Proof. Exercise. □

Remark 2.54. Killing vector fields on induced volume form

Remark 2.55. We have seen the formal definition of isometries between pseudo-Riemannian manifolds and how to describe infinitesimal isometries of a given pseudo-Riemannian manifold. It is, however, in general a very difficult task to verify or disprove that two given pseudo-Riemannian manifolds are isometric. For a reasonable approach to this kind of problem we will need the definitions of the different curvatures of a pseudo-Riemannian manifold, but in order to introduce these we will need to study so-called connections in vector bundles, which is what we will do next.

2.2 Connections in vector bundles

The subject of this chapter, *connections in vector bundles*, is motivated as follows. Suppose that we are given a connected smooth manifold M. We know how to "connect" two points, namely by specifying a smooth curve starting at one and ending at the other. Next, suppose that we are given two tangent vectors $v,w \in TM$ that are contained in different fibres. How do we connect v and w or, more generally, their fibres? Since the total space TM of the tangent bundle is a smooth manifold as well we can of course connect v and w, viewed as points in TM, via a smooth curve. But this does in a sense allow for too much freedom of choice, as we want a in some sense canonical way to *connect* $T_{\pi(v)}M$ with $T_{\pi(w)}M$ via a linear isomorphism. The answer to this problem is to construct a so-called connection in $TM \to M$ with respect to a given pseudo-Riemannian metric, such that all of the latter identifications are linear isometries. Furthermore we require that if we go around an infinitesimal parallelogram in M and consider the identification of tangent spaces, we should end up with the identity. In the following we will in detail describe these concepts and how to actually perform calculations with them. First, we will introduce the most general concept of a connection in a vector bundle and then focus on the tangent bundle and its various tensor bundles.

Remark 2.56. motivation of problem to "connect" tangent spaces at points locally via parallel transport in canonical and polar coordinates in $\mathbb{R}^2$, with picture!

Definition 2.57. Let $E \to M$ be a vector bundle. A **connection in** $E \to M$ is a bilinear map

$$\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E), \quad (X,s) \mapsto \nabla_X s,$$

that is $C^\infty(M)$-linear[32] in the first entry, i.e.

$$\nabla_{fX}s = f\nabla_X s \quad \forall f \in C^\infty(M), X \in \mathfrak{X}(M), s \in \Gamma(E),$$

and fulfils the Leibniz rule

$$\nabla_X(fs) = X(f)s + f\nabla_X s \quad \forall f \in C^\infty(M), X \in \mathfrak{X}(M), s \in \Gamma(E).$$

The last condition can be written as $\nabla(fs) = s \otimes df + f\nabla s$. Note that a connection in $E \to M$ can be canonically extended to local sections.

The defining conditions of a connection hint at their interpretation as certain types of derivatives. In comparison with the Lie derivative (for $E = TM$), we see that it differs from a connection by the tensoriality property in the first argument. Recall that in general $\mathcal{L}_{fX}Y \neq f\mathcal{L}_X Y$ for vector fields $X, Y \in \mathfrak{X}(M)$. This fact points even to preferring a connection over the Lie derivative for the concept of a derivative of sections since a derivative should ideally only depend on the direction in which we are differentiating and the local behaviour of the section we are taking the derivative of, and **not** of the local behaviour of our direction as part of a vector field. Next,we need to ask ourselves how to actually calculate with a connection. The answer lies in the use of local frames.

Definition 2.58. Let ∇ be a connection in $E \to M$ of rank ℓ and $\{s_1, \dots, s_\ell\}$ be a local frame over $U \subset M$ open, such that there exist local coordinates $(x^1, \dots, x^n)$ on $U \subset M$. This can always be achieved after possibly shrinking U. Let further $\dim(M) = n$. Define

$$\nabla s_i := \omega_i, \quad \omega_i(X) = \nabla_X s_i \quad \forall X \in \mathfrak{X}(M),$$

for $1 \leq i \leq \ell$. Then each ω_i is an **E-valued one form**[33], that is $\omega_i \Gamma(E|_U \otimes T^*M|_U)$ for all $1 \leq i \leq \ell$. Thus we have

$$\omega_i = \sum_{j=1}^{n} \omega_{ij} \otimes dx^j$$

for all $1 \leq i \leq \ell$, where $\omega_{ij} \in \Gamma(E|_U)$ for all $1 \leq i \leq \ell$, $1 \leq j \leq n$. We can further write

$$\omega_{ij} = \sum_{k=1}^{\ell} \omega_{ij}^k s_k,$$

with $\omega_{ij}^k \in C^\infty(U)$ for all $1 \leq i \leq \ell$, $1 \leq j \leq n$, $1 \leq k \leq \ell$. Recall that for any local section $s \in \Gamma(E|_U)$ we can write $s = \sum_{i=1}^{k} f^i s_i$ with f^i, $1 \leq i \leq k$, uniquely determined for s. With $X \in \mathfrak{X}(U)$, $X = \sum_{i=1}^{n} X^i \frac{\partial}{\partial x^i}$ we obtain the general formula

$$\nabla_X s = \sum_{i=1}^{\ell} s_i \otimes df^i + \sum_{j=1}^{n} \sum_{i,k=1}^{\ell} f^i \omega_{ij}^k s_k \otimes dx^j. \tag{2.5}$$

On the other hand we might write

$$\nabla s_i = \omega_i = \sum_{k=1}^{\ell} s_k \otimes \omega_i^k$$

for all $1 \leq i \leq k$, where $\omega_i^k \in \Omega^1(U)$ for all $1 \leq i, k \leq \ell$. The ω_i^k are called **connection 1-forms** and determine the connection ∇ in $E|_U$ completely. We might view (ω_i^k) as an $(\ell \times \ell)$-matrix valued map where each entry is a local 1-form on M.

[32] a.k.a. "tensorial"

[33] That means: Plug in a (local) vector field, get a (local) section in E.

Remark 2.59. Warning: Connections, and with them the corresponding connection one forms, do not transform like tensors if E is some tensor power of TM. The reason for that is that a connection itself is not tensorial in the second argument, so changing frames will lead to the new connection one forms to depend on the partial derivatives of the corresponding transformation. The transformation behaviour will be studied in detail for connections in $TM \to M$.

The benefit of writing down a connection locally using its connection 1-form is the easy-to-formulate transformation behaviour when changing the local frame of E (**not** the local coordinates on M). Changing the frame without changing the coordinates on the base space is not too important for our purposes, but nevertheless a nice exercise. Observe in particular that the transformation is not tensorial, that is not simply the pullback in the frame part.

Exercise 2.60. Let ∇ be a connection in a vector bundle $E \to M$ of rank ℓ. Let $\{s_1, \ldots, s_\ell\}$ and $\{\widetilde{s}_1, \ldots, \widetilde{s}_\ell\}$ be local frames of E over a chart neighbourhood $U \subset M$, equipped with local coordinates $(x^1, \ldots, x^n)$, that are related by the $(\ell \times \ell)$-matrix valued smooth map

$$A : U \to \mathrm{GL}(\ell), \quad (s_1, \ldots, s_\ell) \cdot A = \widetilde{s}_1, \ldots, \widetilde{s}_\ell.$$

Let (ω_i^k) denote the matrix of connection 1-forms with respect to the local frame $\{s_1, \ldots, s_\ell\}$ and $(\widetilde{\omega}_i^k)$ the matrix of connection 1-forms with respect to the local frame $\{\widetilde{s}_1, \ldots, \widetilde{s}_\ell\}$. Show that the two matrices of connection 1-forms are related by

$$(\widetilde{\omega}_i^k) = A^{-1} dA + A^{-1} (\omega_i^k) A.$$

In the above equation, dA denotes the differential of the map $A : U \to \mathrm{GL}(\ell)$, where we identify $T\mathrm{GL}(\ell) \cong \mathrm{GL}(\ell) \times \mathrm{End}(\mathbb{R}^\ell)$[34].

Connections, just like tangent vectors, are local objects in the following sense.

Lemma 2.61. Let ∇ be a connection in a vector bundle $E \to M$ of rank ℓ. Let $U \subset M$ be open and suppose that for two vector fields $X, Y \in \mathfrak{X}(M)$ and two sections in $E \to M$, $s, \widetilde{s}$, we have

$$X|_U = Y|_U, \quad s|_U = \widetilde{s}|_U.$$

Then $\nabla_X s$ and $\nabla_Y \widetilde{s}$ coincide on U.

Proof. Note that $\nabla_X s|_U = \nabla_Y s|_U$, which follows by the tensoriality property in the first argument of any connection. It thus suffices to show that $\nabla_X s|_U = \nabla_X \widetilde{s}|_U$. Using Definition 2.58 we write, after possibly shrinking U, s and $\widetilde{s}$ in a local frame $\{s_1, \ldots, s_\ell\}$ of $E|_U$,

$$s|_U = \sum_{i=1}^{\ell} f^i s_i, \quad \widetilde{s}|_U = \sum_{i=1}^{\ell} \widetilde{f}^i s_i,$$

with $f_i, \widetilde{f}_i \in C^\infty(U)$. Now equation (2.5) and $f^i = \widetilde{f}^i$ for all $1 \leq i \leq n$ by assumption that s and $\widetilde{s}$ coincide on U imply that $\nabla_X s|_U = \nabla_X \widetilde{s}|_U$ holds true.

If one prefers to work without coordinates or frames, one can proceed as follows. By the linearity in the second argument, $\nabla_X s$ and $\nabla_X \widetilde{s}$ coincide in U if and only if $\nabla_X (s - \widetilde{s})|_U \equiv 0$. Hence, it suffices to prove $\nabla_X s|_U = 0$ if $s|_U = 0$. Fix $p \in U$ and choose a bump function $b \in C^\infty(M)$ and an open neighbourhood of p, $V \subset U$, that is precompact in U, such that $b|_V \equiv 1$ and $\mathrm{supp}(b) \subset U$. Then by the Leibniz rule

$$0 = \nabla_X 0|_p = \nabla_X (bs)|_p = X(b)s|_p + b(p)\nabla_X s|_p = \nabla_X s|_p.$$

□

[34] Recall that $\mathrm{GL}(\ell)$ is open in the real $(\ell \times \ell)$-matrices.

Lemma 2.61 means that $(\nabla_X s)(p)$ for any $p \in M$ depends only on $X_p \in T_pM$ and the restriction of s to an arbitrary small open neighbourhood of p in M.

Before continuing, we remark that there is a connection we are probably already aware of, although not under that name.

Example 2.62. Consider $\mathbb{R}^n$ with canonical coordinates $(u^1, \dots, u^n)$ and induced global frame $\left\{\frac{\partial}{\partial u^1}, \dots, \frac{\partial}{\partial u^n}\right\}$ of $T\mathbb{R}^n \to \mathbb{R}^n$. Vector fields on $\mathbb{R}^n$ can be, as we described before introducing vector fields on smooth manifolds, viewed as smooth vector valued functions. So a reasonable approach for a connection, defined in our choice of coordinates, is

$$\nabla_X Y := \sum_i X(Y^i)\frac{\partial}{\partial u^i} \in \mathfrak{X}(\mathbb{R}^n)$$

for all vector fields $X = \sum_i X^i \frac{\partial}{\partial u^i}$ and $Y = \sum_i Y^i \frac{\partial}{\partial u^i}$. This means that, in canonical coordinates, we differentiate Y entrywise in X-direction. One verifies that the so-defined ∇ in fact is a connection in $T\mathbb{R}^n \to \mathbb{R}^n$. This construction is, however, not coordinate-independent, meaning that in different coordinates, $\nabla_X Y$ will not be the entrywise differentiation of Y in X-direction. Note that all connection 1-forms of the above connection identically vanish.

As described in the above example, we need to investigate how a connection in TM, written in a choice of local coordinates, behaves under a change of coordinates. This problem is equivalent to understanding how the connection 1-forms transform under a change of coordinates and induces change in local frame of TM. To do so we will introduce so-called Christoffel symbols, which are commonly used to describe connections and, hence, connection 1-forms in the tangent bundle of a smooth manifold. The difference to a connection in a general bundle is that a choice of coordinates on M automatically gives us a local frame in TM.

Definition 2.63. Let ∇ be a connection in $TM \to M$ and $(x^1, \dots, x^n)$ be local coordinates on $U \subset M$. Then in the induced local frame of TM,

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum_{k=1}^n \Gamma^k_{ij} \frac{\partial}{\partial x^k},$$

where $\Gamma^k_{ij} \in C^\infty(M)$, $1 \leq i, j, k \leq n$. The terms Γ^k_{ij} are called **Christoffel**[35] **symbols** of the connection ∇ with respect to the chosen local coordinates $(x^1, \dots, x^n)$. The Christoffel symbols specify the connection ∇ in $TM|_U \to U$ uniquely, meaning in particular that two connections in $TM \to M$ coincide if they have the same Christoffel symbols for all local coordinates on M. In comparison with the most general case, the Christoffel symbols are for the special case of the tangent bundle with induced local frame precisely the terms ω^k_{ij} in equation (2.5).

Note that, if one wants to be very precise it is at this point not clear if every manifold admits a connection in its tangent bundle. This is either a not so easy exercise or a good excuse to consult [L1, Prop. 4.5]. The answer is yes, every manifolds admits a connection in its tangent bundle, and the space of connections is, in a sense, very big.

Similar to Exercise 2.60, but with the difference that we now also change the local coordinates on the base manifold, we obtain the following transformation ruse for Christoffel symbols.

Lemma 2.64. Let M be an n-dimensional smooth manifold, ∇ a connection in $TM \to M$. Let further $\varphi = (x^1, \dots, x^n)$ and $\psi = (y^n, \dots, y^n)$ local coordinate systems on an open set $U \subset M$ so that

$$F(y^1, \dots, y^n) = (x^1, \dots, x^n)$$

[35] Elwin Bruno Christoffel (1829 – 1900)

for a smooth map $F : \psi(U) \to \varphi(U)$, and let Γ^k_{ij} denote the Christoffel symbols of ∇ with respect to φ and $\widetilde{\Gamma}^k_{ij}$ denote the Christoffel symbols of ∇ with respect to ψ. Then the following identity holds:

$$\Gamma^k_{ij} =$$

Proof. Direct calculation using $\frac{\partial}{\partial x^i} = \sum_j \frac{\partial F^i}{\partial y^j} \frac{\partial}{\partial y^j}$ and the corresponding inverse formula. ☐

Suppose that we are given a connection ∇ in $TM \to M$. Then ∇ induces a connection in all tensor powers $T^{r,s}M \to M$ of the tangent bundle by requiring compatibility with contractions.

Lemma 2.65. Let ∇ be a connection in $TM \to M$. Then ∇ induces a connection in each tensor bundle[36] $T^{r,s}M \to M$, $r \geq 0$, $s \geq 0$, such that

(i) the induced connection in $T^{1,0}M \cong TM \to M$ coincides with ∇,

(ii) $\nabla f = df$ for all $f \in \mathcal{T}^{0,0}(M) = C^\infty(M)$,

(iii) the induced connection is a tensor derivation in the second argument, meaning that

$$\nabla(A \otimes B) = (\nabla A) \otimes B + A \otimes (\nabla B)$$

whenever the tensor field $A \otimes B$ is defined,

(iv) the induced connections commute with all possible contraction, meaning that for any contraction $C : \mathcal{T}^{r,s}(M) \to \mathcal{T}^{r-1,s-1}(M)$ we have

$$\nabla(C(A)) = C(\nabla(A))$$

for all tensor fields $A \in \mathcal{T}^{r,s}(M)$.

The so-defined connections in each tensor bundle $T^{r,s}M \to M$ are uniquely determined by the above properties.

Proof. We proceed as follows. First we define a candidate for a connection, then we show that it fulfils all of the above properties, and finally prove uniqueness. In order to define any connection in $T^{r,s}M \to M$ it suffices to specify what it does on sections that can be, locally, written as pure tensor products of r local vector fields and s local 1-forms. For $T^{1,0}M \to M$, we simply take ∇ to be our initial connection, which thereby automatically fulfils (i) and for $f \in \mathcal{T}^{0,0}(M) = C^\infty(M)$ we set $\nabla f = df$, thereby fulfilling (ii). Now we define ∇ in $T^{0,1}M \to M$ in such a way, that (iii) and (iv) will be satisfied. Set for any local 1-form $\omega \in \Omega^1(U)$, $U \subset M$ open,

$$(\nabla_X \omega)(Y) := X(\omega(Y)) - \omega(\nabla_X Y)$$

for all local vector fields $X, Y \in \mathfrak{X}(U)$. After checking [Exercise!] that this defines a connection in $T^{0,1}M \to M$, we proceed as initially mentioned and obtain a connection in $T^{r,s}M \to M$ for all $r \geq 0$, $s \geq 0$ by requiring (ii) to hold on pure and, hence by linear extension, on all tensor fields. After checking that this really does define a connection [Again, exercise!] it remains to check that (iv) holds. This can be done inductively using (iii) after checking that it holds for the only possible contraction in $T^{1,1}M \to M$, which on pure tensor fields is of the form

$$C(X \otimes \omega) = \omega(X)$$

[36] Conventionally denoted by the same symbol ∇.

for all $X \in \mathfrak{X}(M)$, $\omega \in \Omega^1(M)$, and analogously for local sections. We find for all $X, Y \in \mathfrak{X}(M)$ and all $\omega \in \Omega^1(M)$

$$\nabla_Y(C(X \otimes \omega)) = \nabla_Y(\omega(X)) = Y(\omega(X))$$

which by definition of ∇ in $T^{0,1}M \to M$ and our imposed condition (iii) coincides with

$$Y(\omega(X)) = (\nabla_Y \omega)(X) + \omega(\nabla_Y X) = C(X \otimes (\nabla_Y \omega) + (\nabla_Y X) \otimes \omega = C(\nabla_Y(X \otimes \omega)).$$

It remains to show uniqueness. Suppose there is an other connection $\tilde{\nabla}$ fulfilling all requirements of this lemma. By linearity in the second argument it suffices to show that ∇ and $\tilde{\nabla}$ coincides on locally pure tensor fields. By (i) and (iii) it further suffices to show that ∇ and $\tilde{\nabla}$ coincide in $T^{0,1}M = T^*M \to M$. This follows from (i), (ii), and (iv) by direct calculation of the left- and right-hand of $\tilde{\nabla}(C(A)) = C(\tilde{\nabla}(A))$ for $A = X \otimes \omega$ where X is any local vector field and ω is any local 1-form. □

Observe that Lemma 2.65 is, formally, very similar to Proposition 1.156 about the Lie derivative of tensor fields.

Exercise 2.66. Let ∇ be a connection in $TM \to M$ and Γ_{ij}^k its Christoffel symbols in local coordinates $(x^1, \dots, x^n)$. Find a formula for the Christoffel symbols of the induced connection ∇ in $T^*M \to M$ with respect to the local frame obtained from the local coordinate 1-forms.

Remark 2.67. Differentiation of tensor fields with respect to a connection induced by a connection in the tangent bundle is sometimes called **covariant differentiation**. $\nabla_X A$ is then called **covariant derivative of A in direction** X. When talking about covariant derivatives make sure to always specify the corresponding connection.

A central usage of connections is a preferred way to "transport", that is smoothly change, vectors along a curve in the base manifold. In order to properly introduce this concept, we need to study how to in a covariant manner differentiate vector fields, or more generally tensor fields, along curves. This is to be read in the way that we want to give a meaning to expressions of the form

$$\nabla'_\gamma A$$

where $\gamma : I \to M$ is a smooth curve in a manifold and A is a tensor field that is **only defined along** $\gamma(I) \subset M$. Recall in particular the definition of vector fields along curves, cf. Definition 1.101.

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Todo list

Put exercises at the end of each chapter, not necessarily from exercise sheets.