Universität Hamburg

Department Mathematik
Analysis und Differentialgeometrie

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Differential geometry **summer term 2020**

Exercise sheet 9

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1. (8 points)
Let (M, g) be a pseudo-Riemannian manifold and ∇ its Levi-Civita connection. For a function $F \in C^\infty(M)$, the gradient $\mathrm{grad}_g(F) \in \mathfrak{X}(M)$ of F with respect to g is defined to be the unique vector field, such that

$$dF(X) = g(\mathrm{grad}_g(F), X) \ \forall X \in \mathfrak{X}(M).$$

Let $f \in C^\infty(M)$, $f(p) > 0$ for all $p \in M$. Consider the pseudo-Riemannian manifold $(M, \widetilde{g})$ with $\widetilde{g} = fg$ and Levi-Civita connection $\widetilde{\nabla}$ ($\widetilde{g}$ is called a conformal rescaling of g). Show that

$$\widetilde{\nabla}_X Y - \nabla_X Y = \frac{1}{2}\left(d\ln(f)(X)Y + d\ln(f)(Y)X - g(X,Y)\mathrm{grad}_g(\ln f)\right) \ \forall X, Y \in \mathfrak{X}(M),$$

where $\mathrm{grad}_g(\ln(f))$ is calculated with respect to g. Conclude that for a constant rescaling $f = \text{const.}$, g and $\widetilde{g}$ induce the same Levi-Civita connection. *(Hint: Do not use Christoffel symbols to solve this exercise.)*

2. (4 points)
Let (M, g) be a pseudo-Riemannian manifold and ∇ its Levi-Civita connection. Show that the covariant Hessian of a smooth function $f \in C^\infty(M)$, $H^f := \nabla^2 f \in \mathcal{T}^{0,2}(M)$, w.r.t. ∇ is symmetric and that

$$H^f(X, Y) = g(\nabla_X(\mathrm{grad}_g(f)), Y)$$

for all $X, Y \in \mathfrak{X}(M)$.

3. (4 points)
Let ∇ be a connection in $TM \to M$ and Γ_{ij}^k its Christoffel symbols in local coordinates $(x^1, \ldots, x^n)$. Find a formula for the analogue of the Christoffel symbols of the induced connection ∇ in $T^*M \to M$ with respect to the local frame given by the local coordinate 1-forms.