Universität Hamburg

Department Mathematik
Analysis und Differentialgeometrie

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Differential geometry **summer term 2020**

Exercise sheet 8

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1. (4 points)
Let ∇ be a connection in the tangent bundle of a manifold M, $\dim M \geq 1$. Show that the torsion tensor of the connection ∇, $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$, is, in fact, a tensor field. Show that $T = 0$ if and only if the Christoffel symbols of ∇ have the property that $\Gamma^k_{ij} = \Gamma^k_{ji}$ for all $1 \leq i,j,k \leq \dim M$.

2. (4 points)
Let M be a manifold and let ∇ and $\overline{\nabla}$ be two connection in $TM \to M$. We define $A(X,Y) := \nabla_X Y - \overline{\nabla}_X Y$ for all $X, Y \in \mathfrak{X}(M)$. Show that

(a) A is a tensor field,

(b) A is symmetric, that is $A(X,Y) = A(Y,X)$ for all $X, Y \in \mathfrak{X}(M)$, if and only if the torsion tensors of ∇ and $\overline{\nabla}$ coincide.

3. (4 points)
Prove or disprove: For a connection ∇ in $TM \to M$, the induced connection in $T^{0,2}M \to M$ restricts to a connection in the subbundles

a) $\mathrm{Sym}^2(T^*M) \to M$,

b) $\Lambda^2 T^*M \to M$.

4. (4 points)
Let (M,g) be a pseudo-Riemannian manifold of dimension $n \geq 1$. Show that the Koszul formula is equivalent to

$$2\Gamma^k_{ij} = \sum_{l=1}^{n} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}) g^{lk} \text{ for all } 1 \leq i,j,k \leq n$$

for all local coordinates $(x^1, \cdots, x^n)$ of M.