Universität Hamburg

Department Mathematik
Analysis und Differentialgeometrie

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Differential geometry **summer term 2020**

Exercise sheet 7

— submission deadline: 19.6.2020 —

1. (4 points)
Show that the set of Killing vector fields on a pseudo-Riemannian manifold (M, g) is a Lie subalgebra of $(\mathfrak{X}(M), [\cdot, \cdot])$.

2. (9 points)
Prove the statements in Example 2.49 in the lecture notes, that is show that the following actually **are** Killing vector fields:

(i) Let A be an $(n+1)\times(n+1)$ skew real matrix, that is $A^T = -A$. Then $e^{At} \in \mathrm{O}(n+1)$ for all $t \in \mathbb{R}$. Then the vector field $X \in \mathfrak{X}(S^n)$ given by

$$X_p = \left.\frac{\partial}{\partial t}\right|_{t=0} \left(e^{At}p\right) \in T_pS^n$$

is a Killing vector field of the standard round metric on S^n, that is the restriction of the pointwise Euclidean scalar product in the ambient manifold $\mathbb{R}^{n+1}$. Note that $e^{A\cdot} : \mathbb{R} \times S^n \to \mathbb{R}^{n+1}$, $(t, v) \mapsto e^{At}v$ is the global flow of X.

(ii) Consider $(\mathbb{R}^n, \langle\cdot,\cdot\rangle_\nu)$ for any $0 \leq \nu \leq n$ and fix $(c^1, \ldots, c^n) \in \mathbb{R}^n$. Then $X \in \mathfrak{X}(\mathbb{R}^n)$, $X = \sum_i c^i \frac{\partial}{\partial u^i}$, is a Killing vector field.

(iii) Let (M, g) and (N, h) be pseudo-Riemannian manifolds, X a Killing vector field on (M, g), and Y a Killing vector field on (N, h). Then $X + Y$ is a Killing vector field on $(M \times N, g \oplus h)$.

3. (6 points)
Prove Lemma 2.50 in the lecture notes: Prove that $X \in \mathfrak{X}(M)$ on a pseudo-Riemannian manifold (M, g) is a Killing vector field if and and only if it fulfils

$$\sum_{k=1}^{n} \left(X^k \frac{\partial g_{ij}}{\partial x^k} + \frac{\partial X^k}{\partial x^i} g_{jk} + \frac{\partial X^k}{\partial x^j} g_{ik} \right) = 0 \quad \forall 1 \leq i, j \leq n$$

for all local coordinates $(x^1, \ldots, x^n)$ on M.

4. (Bonus: 11 points)
Define the bundles of symmetric and antisymmetric $(0, k)$-tensors as subbundles of $T^{0,k}M \to M$, that is $\mathrm{Sym}^k(T^*M) \to M$ and $\Lambda^k(T^*M) \to M$, for all $k \geq 3$. Determine their rank for each k.