Universität Hamburg

Department Mathematik
Analysis und Differentialgeometrie

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Differential geometry **summer term 2020**

Exercise sheet 6

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1. (4 points)
Consider the real vector space $\mathbb{R}^2$ equipped with the bilinear forms that are given in the standard basis by

$$g(v,v) = \begin{pmatrix} v^1 \\ v^2 \end{pmatrix}^T \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \end{pmatrix}, \ a \in \mathbb{R}.$$

Determine all $a \in \mathbb{R}$, such that g is degenerate. Determine the index of g for all $a \in \mathbb{R}$ for which g is non-degenerate.

2. (8 points)

(a) Let V be a finite dimensional real vector space. We define $S^2V^* := \{\alpha \in V^* \otimes V^* \mid \alpha(v,w) = \alpha(w,v)\ \forall v,w \in V\}$ ("S^2"means symmetric in the 2 arguments) and $V^* \wedge V^* := \{\alpha \in V^* \otimes V^* \mid \alpha(v,w) = -\alpha(w,v)\ \forall v,w \in V\}$. Show that $V^* \otimes V^* = S^2V^* \oplus (V^* \wedge V^*)$.

(b) Consider $\mathbb{R}^3$ as a manifold equipped with the pseudo-Riemannian metric g given in standard coordinates (x,y,z) by

$$g_p(v,w) = v^T \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} w, \quad \forall p \in \mathbb{R}^3, v,w \in T_p\mathbb{R}^3 \cong \mathbb{R}^3.$$

The set $H := \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 = -1\}$ is a smooth submanifold of $\mathbb{R}^3$. Draw a sketch of H and show that $(H, g|_{TH \times TH})$ is a Riemannian manifold. $g|_{TH \times TH}$ denotes the restriction of g to TH, so that for $v,w \in T_pH$, $g|_{TH\times TH}(v,w) = g_p(v,w)$, where we view TH as a subset of $T\mathbb{R}^3$: $T_pH = \ker d(x^2+y^2-z^2+1)_p \subset T_p\mathbb{R}^3$ for all $p \in H$.

3. (4 points)
Let (M,g) be a connected pseudo-Riemannian manifold of dimension $n \geq 1$. We define

$$\mathrm{index}(g_p) = \max\{\dim W \mid W \subset T_pM \text{ linear subspace, such that } g_p|_{W\times W} \text{ is negative definite}\}.$$

(a) For $p \in M$, we choose local coordinates $(x^1, \ldots, x^n)$ and define $B \in \mathrm{Mat}(n \times n)$, $B_{ij} = g_p\left(\left.\frac{\partial}{\partial x^i}\right|_p, \left.\frac{\partial}{\partial x^j}\right|_p\right)$. Show that

$$\mathrm{index}(g_p) = |\{\text{negative eigenvalues of } B \text{ counted with multiplicity}\}|.$$

(b) Show that this definition does not depend on the choice of local coordinates around $p \in M$. *(Hint: If you have difficulties with this part write down how the matrix B transforms under a change of coordinates.)*

4. (4 points)
Consider the manifold $\mathrm{GL}(n) \subset \mathrm{Mat}(n \times n, \mathbb{R})$ together with the $(0,2)$-tensor field $g \in \mathcal{T}^{0,2}(\mathrm{GL}(n))$ that is given in canonical coordinates by

$$g_A(B, C) = \mathrm{tr}(B^T C) \quad \forall A \in \mathrm{GL}(n),\ B, C \in T_A\mathrm{GL}(n) = \mathrm{Mat}(n \times n, \mathbb{R}),$$

where tr denotes the trace, i.e. $\mathrm{tr}(F) = \sum_{i=1}^{n} F_{ii}$ for all $F \in \mathrm{Mat}(n \times n, \mathbb{R})$. Show that $(\mathrm{GL}(n), g)$ is a Riemannian manifold.

5. (Bonus: 8 points)
Let M be a manifold of dimension $n \geq 1$. Show that M admits a Riemannian metric. *(Hint: Assume that $\langle \cdot, \cdot \rangle$ and $(\cdot, \cdot)$ are Euclidean scalar products on $\mathbb{R}^n$. Is $\langle \cdot, \cdot \rangle + (\cdot, \cdot)$ also a Euclidean scalar product? You can also assume without proof that M has a countable atlas, which follows from the 2nd countability property of M. Furthermore, you might want to look into exhaustion by compact sets of non-compact manifolds.)*