Universität Hamburg

Department Mathematik
Analysis und Differentialgeometrie

David Lindemann
Danu Thung

Differential geometry **summer term 2020**

Exercise sheet 5

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1. (8 points)
Consider the 2-torus $T^2 = \mathbb{R}^2/\sim$ as it was defined in the first exercise sheet. Denote by $\rho : \mathbb{R}^2 \to T^2$ the projection map $p \mapsto [p]$. For a fixed paramater $s \in \mathbb{R}$, let $\overline{X} \in \mathfrak{X}(\mathbb{R}^2)$, given in canonical coordinates (x, y) by $\overline{X} = \frac{\partial}{\partial x} + s\frac{\partial}{\partial y}$. Consider $X \in \mathfrak{X}(T^2)$ defined by $X = d\rho(\overline{X})$. The flow of X is given by

$$\varphi_t([p]) = \left[p + t\begin{pmatrix}1\\ s\end{pmatrix}\right]$$

and it is defined for all $t \in \mathbb{R}$. Show that for any fixed $[p] \in T^2$

(a) $s \in \mathbb{Q}$ implies that $t \mapsto \varphi_t([p])$ is periodic,

(b) $s \in \mathbb{R} \setminus \mathbb{Q}$ implies that $t \mapsto \varphi_t([p])$ is not periodic,

(c) $s \in \mathbb{R} \setminus \mathbb{Q}$ implies that the image of the flow $\varphi_{\mathbb{R}}([p]) \subset T^2$ is dense for all $[p] \in T^2$, that is $\overline{\varphi_{\mathbb{R}}([p])} = T^2$. In this case this means that for all $q \in T^2$ and all open neighbourhoods $U \subset T^2$ of q there exists a $t' \in \mathbb{R}$, such that $\varphi_{t'}([p]) \in U$ (recall that U is open in T^2 if $\rho^{-1}(U) \subset \mathbb{R}^2$ is open).

2. (4 points)
Consider for $n \geq 1$ fixed the n-dimensional open unit ball around the origin $B_1(0) \subset \mathbb{R}^n$. For a given $v \in \mathbb{R}^n$ consider the vector field $X \in \mathfrak{X}(B_1(0))$ given in canonical coordinates $(u^1, \cdots, u^n)$ by $X = \sum_{i=1}^{n} v^i \frac{\partial}{\partial u^i}$. Determine the flow $\varphi_{(\cdot)}(p) : (-\varepsilon_p, \varepsilon_p) \to B_1(0)$ of X for p arbitrary but fixed and $\varepsilon_p > 0$ small enough. Show that

(a) $v \neq 0$ implies $\inf_{p \in B_1(0)} \{ \sup_{s \in \mathbb{R}_{\geq 0}} \{ s > 0 \mid \varphi_{(\cdot)}(p) : [0, s) \to B_1(0) \text{ is defined} \} \} = 0$,

(b) $v = 0$ implies $\varphi_{(\cdot)}(p) : (-\varepsilon_p, \varepsilon_p) \to B_1(0)$ can be extended to $\mathbb{R}$ for all $p \in B_1(0)$.

3. (4 points)
Let $M = \mathbb{R}^2$ and $\alpha \in \mathcal{T}^{(1,2)}(M)$, $\alpha = y^2 \frac{\partial}{\partial x} \otimes dx \wedge dy$, where $dx \wedge dy := dx \otimes dy - dy \otimes dx$. Let $X \in \mathfrak{X}(\mathbb{R}^2)$, $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$. Calculate $\mathcal{L}_X \alpha$.