Universität Hamburg

Department Mathematik
Analysis und Differentialgeometrie

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Differential geometry **summer term 2020**

Exercise sheet 4

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1. (4 points)

(a) Let $M = \mathbb{R}^2 \setminus \{(z,0) \mid z \in \mathbb{R}_{\leq 0}\}$ and $X \in \mathfrak{X}(M)$. Assume that in Cartesian coordinates (x,y), X is given by $X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$. Find the formula for X in polar coordinates.

(b) Let $M = \mathbb{R}^n$, $n \geq 2$, and let $N \subset M$ be an $(n-1)$-dimensional submanifold, such that $(M \setminus N) \subset M$ is an n-dimensional submanifold. Let $Y \in \mathfrak{X}(M \setminus N)$ be an arbitrary vector field.

i) Can you always find a vector field $X \in \mathfrak{X}(M)$, such that $X|_{M\setminus N} = Y$?

ii) Let $X_1, X_2 \in \mathfrak{X}(M)$, such that $X_1|_{M\setminus N} = X_2|_{M\setminus N}$. Show that this implies $X_1 = X_2$.

2. (4 points)

(a) Let M be a smooth manifold and $X, Y \in \mathfrak{X}(M)$. Find a formula for $[X,Y]$ in local coordinates.

(b) Show that for fixed $0 \neq X \in \mathfrak{X}(M)$, the map $[X,\cdot] : \mathfrak{X}(M) \to \mathfrak{X}(M)$, $Y \mapsto [X,Y]$, is a vector space homomorphism, but it is not a $C^\infty(M)$-module homomorphism [heuristically speaking: "$\mathbb{R}$-linear but not $C^\infty(M)$-linear"].

(c) Prove or disprove the following: There exists a manifold M of dimension $n \geq 1$, such that for all $0 \neq X \in \mathfrak{X}(M)$, the map $[X,\cdot] : \mathfrak{X}(M) \to \mathfrak{X}(M)$ is surjective.

3. (4 points)
Let $A \in \mathrm{GL}(n)$ be fixed. Since $\mathrm{GL}(n) \subset \mathrm{Mat}(n \times n, \mathbb{R}) \cong \mathbb{R}^{n^2}$ is open we can identify $T\mathrm{GL}(n) = \mathrm{GL}(n) \times \mathrm{Mat}(n \times n, \mathbb{R})$. We define $X \in \mathfrak{X}(\mathrm{GL}(n))$ by $X_B = (B, AB)$, where $B \in \mathrm{GL}(n)$ and $AB \in \mathrm{Mat}(n \times n, \mathbb{R})$. With this identification, a curve $\gamma : I \to \mathrm{GL}(n)$ is an integral curve of X if $A\gamma(t) = \gamma'(t)$ for all $t \in I$. Find the general solution of this equation for all $A \in \mathrm{GL}(n)$ and show that each local solution γ can be extended to $\gamma : \mathbb{R} \to \mathrm{GL}(n)$.

4. (4 points)
Let $S^3 \subset \mathbb{H} \cong \mathbb{R}^4$ be the 3-sphere, viewed as the set of unit quaternions. Show that TS^3 is trivializable.