Universität Hamburg

Department Mathematik
Analysis und Differentialgeometrie

David Lindemann
Danu Thung

Differential geometry **summer term 2020**

Exercise sheet 3

— submission deadline: 15.5.2020 —

---

1. (8 points)
Show that every vector bundle E of rank $k \geq 1$ over a manifold M with projection map $\pi : E \to M$ has a non-trivial section $s \in \Gamma(E)$, i.e. there exists $p \in M$, such that $s(p) \neq 0 \in \pi^{-1}(p)$.

Is it also true that we can always find a section $\eta \in \Gamma(E)$, such that $\eta(p) \neq 0$ for all $p \in M$? *(Hint: Recall orientability of $(n-1)$-dimensional submanifolds of $\mathbb{R}^n$ and think of implications if the latter statement was true.)*

2. (4 points)
Let M be a compact smooth manifold. Assume that $\dim M \geq 1$ and let $f \in C^\infty(M)$ be an arbitrary smooth function. Show that f has at least 2 critical points.

3. (4 points)

a) Show that S^n is an embedded submanifold of S^{n+1} for all $n \in \mathbb{N}$. [Note: First write down an embedding.]

b) Show that the total space of the vector bundle

$$S^n \times \mathbb{R} \to S^n$$

is diffeomorphic to $\mathbb{R}^{n+1} \setminus \{0\}$.

c) Bonus[1]: Let T^2 be the 2-Torus. Prove (and sketch!) that the total space of the vector bundle

$$T^2 \times \mathbb{R} \to T^2$$

is diffeomorphic to $\mathbb{R}^3 \setminus (S^1 \sqcup \mathbb{R})$. Here, S^1 is embedded in the x-y-plane so that $S^1 = \{x^2 + y^2 = 1,\ z = 0\} \subset \mathbb{R}^3$ and $\mathbb{R} = \{x = y = 0\} \subset \mathbb{R}^3$.

[1] I just made this up because it “looks” right in my head. If you can disprove the statement of **3.**c) I will buy you a coffee once the lockdown is over. If you can prove it instead I will also buy you a coffee.