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Analysis und Differentialgeometrie

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Differential geometry **summer term 2020**

Exercise sheet 2

— submission deadline: 8.5.2020 —

1. (4 points)
Let $M = (0, 1) \times (0, 1) \subset \mathbb{R}^2$. Find (and sketch) a smooth injective immersion $\psi : M \to \mathbb{R}^3$, such that

(a) $\psi(M) \subset \mathbb{R}^3$ is closed and not contained in any compact subset of $\mathbb{R}^3$,

(b) $\psi(M) \subset \mathbb{R}^3$ is contained in some compact set $K \subset \mathbb{R}^3$, is not compact, and the boundary of $\psi(M)$ is not connected,

(c) $\psi(M) \subset \mathbb{R}^3$ is compact (a sketch is sufficient).

2. (4 points)
Consider the smooth manifolds $S^n \subset \mathbb{R}^{n+1}$ and $\mathbb{R}P^n = (\mathbb{R}^{n+1} \setminus \{0\})/_\sim$ with their respective smooth structures defined in the lecture notes for $n \in \mathbb{N}$ (via the stereographic projections and the so-called inhomogeneous coordinates, respectively).

(a) Show that for all $n \in \mathbb{N}$ the projection map $\pi : S^n \to \mathbb{R}P^n$, $x \mapsto \mathbb{R}x$, is smooth and a local diffeomorphism near every point $p \in S^n$.

(b) Show that for all $n \in \mathbb{N}$ the projection map $\pi : S^n \to \mathbb{R}P^n$ is not a diffeomorphism. Find a map $\eta : S^1 \to \mathbb{R}P^1$, such that η is a diffeomorphism.

3. (8 points)
In the following, "$\cong$" means isomorphic as vector-spaces and as real smooth manifolds.

(a) Consider $S^1 \subset \mathbb{C}$ as the set of unit complex numbers endowed with the submanifold structure[1] induced by the ambient space $\mathbb{C} \cong \mathbb{R}^2$. Determine all $r \in \mathbb{Z}$, such that the map $f_r : S^1 \to S^1$, $z \mapsto z^r$, is

i) smooth,

ii) a local diffeomorphism,

iii) a diffeomorphism.

[Also think about the following: Would f_r still be well-defined if we allowed r to be an element of $\mathbb{R} \setminus \mathbb{Z}$?]

(b) Show that

$$\mathrm{GL}(n) = \{A \in \mathrm{Mat}(n \times n, \mathbb{R}) \mid A \text{ is invertible}\}$$

is a smooth submanifold of $\mathbb{R}^{n^2} \cong \mathrm{Mat}(n \times n, \mathbb{R})$ and determine its dimension. Show that the map $\iota : \mathrm{GL}(n) \to \mathrm{GL}(n)$, $\iota(A) = A^{-1}$, is smooth and determine its differential.

[1] Note: S^1 as described here is diffeomorphic to S^1 as in the lecture notes with atlas given by the stereographic projections.