Universität Hamburg

Department Mathematik
Analysis und Differentialgeometrie

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Differential geometry **summer term 2020**

Exercise sheet 1

— submission deadline: 1.5.2020 —

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1. (4 points)

(a) Show that the 2-torus $T^2 := \mathbb{R}^2/\sim$,

$$(x,y) \sim (x',y') :\Leftrightarrow \exists z_1, z_2 \in \mathbb{Z} : \; x' = x + z_1, \; y' = y + z_2,$$

is a smooth manifold, i.e. can be endowed with a smooth atlas.

(b) Draw a sketch of T^2 and the charts of an atlas with precisely 3 charts.

2. (4 points)
Let M be a compact smooth manifold with maximal atlas $\overline{\mathcal{A}}$.

(a) Show that there exists a finite atlas $\mathcal{A}$ on M that is equivalent to $\overline{\mathcal{A}}$.

(b) Let $\mathcal{A} = \{(U_i, \varphi_i) \mid i \in I\}$, $|I| < \infty$, be a finite atlas for M. Prove the existence of a partition of unity subordinate to $\mathcal{A}$. A partition of unity subordinate to $\mathcal{A}$ is a set of smooth functions $\{h_i : M \to \mathbb{R}_{\geq 0} \mid i \in I\}$, such that

$$\sum_{i \in I} h_i(p) = 1 \quad \forall p \in M$$

and

$$\operatorname{supp}(h_i) = \overline{\{p \in M \mid h_i(p) \neq 0\}} \subset U_i \quad \forall i \in I,$$

with the requirement that $\operatorname{supp}(h_i)$ is compact for all $i \in I$.

3. (4 points)
Consider the set $M := \{(x,x) \in \mathbb{R}^2 \mid x \in \mathbb{R}\} \cup \{(x,-x) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$ with the induced subspace topology. Show that M cannot be equipped with the structure of a smooth manifold, that is there exists no equivalence class of n-dimensional C^∞-atlases on M for all $n \in \mathbb{N}$. *(Remark: Recall that $U' \subset M$ is open if and only if there exists an open set $U \subset \mathbb{R}^2$, such that $U' = M \cap U$. Consider the properties of the connected components of $M \setminus \{0\}$. You can also use that $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic for $n \neq m$.)*

4. (4 points)
Find an example of two manifolds M and N, both of dimension at least 1, with maximal atlases $\mathcal{A}_M$ and $\mathcal{A}_N$, respectively, so that the product atlas $\mathcal{A}_M \times \mathcal{A}_N$ on $M \times N$ is not maximal. Is the statement that the product atlas is never maximal true for all M, N, with $\dim(M) \geq 1$, $\dim(N) \geq 1$?