Universität Hamburg

Department Mathematik
Analysis und Differentialgeometrie

David Lindemann
Danu Thung

**Differential geometry** **summer term 2020**

# Exercise sheet 11

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**1. (4 points)**
Let $M$ be an $n$-dimensional manifold, $n \geq 1$. Fix a point $p \in M$ and consider the set of abstract curvature tensors
$\{F : T_pM \times T_pM \times T_pM \to T_pM \mid F \text{ is an abstract curvature tensor}\} \subset T_p^{1,3}M$. Show that they form a real vector space and determine its dimension in dependence of $n$.

**2. (4 points)**
Let $\nabla$ be a connection in a vector bundle $E \to M$. For an $E$**-valued 2-form** $\alpha \in \Omega^2(M,E)$, that is a section $\alpha \in \Gamma(E \otimes T^{0,2}M)$, $\alpha : (X,Y) \mapsto \alpha(X,Y) \in \Gamma(E)$, such that $\alpha(X,Y) = -\alpha(Y,X)$ for all $X, Y \in \mathfrak{X}(M)$, we define

$$d^\nabla \alpha(X,Y,Z) := \sum_{\text{cyclic}} (\nabla_X(\alpha(Y,Z)) - \alpha([X,Y],Z)) \quad \forall X,Y,Z \in \mathfrak{X}(M).$$

(i) Show that $d^\nabla \alpha \in \Omega^3(M,E)$, that is an $E$**-valued 3-form**. This means that $d^\nabla\alpha(X,Y,Z) = -d^\nabla\alpha(Y,X,Z)$, $d^\nabla\alpha(X,Y,Z) = -d^\nabla\alpha(X,Z,Y)$, and $d^\nabla\alpha(X,Y,Z) = -d^\nabla\alpha(Z,Y,X)$ for all $X,Y,Z \in \mathfrak{X}(M)$.

(ii) Recall that a connection in $TM \to M$ induces a connection in $\mathrm{End}(TM) \to M$. The curvature tensor[1] of $\nabla$, $R := (X,Y) \mapsto \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$ for all $X,Y \in \mathfrak{X}(M)$, can be viewed as a $\mathrm{End}(TM)$-valued 2-form $R \in \Omega^2(M, \mathrm{End}(E))$. $R$ fulfils

$$\sum_{\text{cyclic}} \nabla_X(R(Y,Z)) = 0$$

for all $X,Y,Z \in \mathfrak{X}(M)$ with $[X,Y] = [Y,Z] = [Z,X] = 0$. Show that this implies $d^\nabla R = 0$.

**3. (4 points)**
Consider a curve $\gamma : I \subset \mathbb{R} \to \mathbb{R}^2$, $I \subset \mathbb{R}$ an open interval. Assume that $\gamma = (\gamma_1, \gamma_2)^T$ is also an embedding, which implies that $\gamma(I) \subset (\mathbb{R}^2, \langle\cdot,\cdot\rangle)$ is a Riemannian submanifold ($\langle\cdot,\cdot\rangle$ denotes the standard Riemannian metric on $\mathbb{R}^2$). Check that the vector field $\xi = (-\dot\gamma_2, \dot\gamma_1)$ along $\gamma$ is orthogonal to $\dot\gamma$. Show that the Weingarten map $S^\xi$ is of the form $S^\xi_{\gamma(t)} = f(\gamma(t)) \cdot \mathrm{Id}_{T\gamma(I) \subset T\mathbb{R}^2|_{\gamma(I)}}$, where $f \circ \gamma$ is a smooth function on $I$, and determine $f \circ \gamma$. Conclude that $S^\xi_{\gamma(t)} \neq 0$ if and only if $\dot\gamma(t)$ and $\ddot\gamma(t)$ are linearly independent. *(Hint: If you are not sure where to start, try rewriting the equation $0 = \overline{\nabla}_{\dot\gamma}(\langle\dot\gamma,\xi\rangle)$, where $\overline{\nabla}$ denotes the flat connection on $\mathbb{R}^2$.)*

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[1]Note: We do **not** require $R$ to be the Riemann curvature tensor of some metric.

**4. (4 points)**
Recall the definition of the hyperboloids $H^n_\nu$ as pseudo-Riemannian submanifolds of index $\nu - 1$ in $(\mathbb{R}^{n+1}, \langle\cdot,\cdot\rangle_\nu)$. Show that the position vector field $\xi \in \mathfrak{X}(\mathbb{R}^{n+1})$ is a unit normal along $H^n_\nu$ and prove that the corresponding Weingarten map is given by $S^\xi = -\mathrm{id}_{TH^n_\nu}$.

**5. (12 (4+8) points)**

(a) Similarly to $H^n_\nu$ we define for $r > 0$, $n \geq 2$, and $1 \leq \nu \leq n$

$$H^n_\nu(r) := \left\{ \langle\xi,\xi\rangle = \sum_{i=1}^{n-\nu+1} \left(u^i\right)^2 - \sum_{i=n-\nu+2}^{n+1} \left(u^i\right)^2 = -r^2 \right\} \subset \mathbb{R}^{n+1},$$

viewed as pseudo-Riemannian submanifolds of $(\mathbb{R}^{n+1}, \langle\cdot,\cdot\rangle_\nu)$. Show that $H^n_1(r)$ has constant sectional curvature $K = -\frac{1}{r^2}$.

(b) Prove that $H^n_1(r)$ is geodesically complete.

**6. (12 points)**
Let $U \subset \mathbb{R}^2$ be open and connected, and $F : U \to \mathbb{R}$ a smooth map. Consider the graph of $F$, $M := \{(x, y, F(x, y)) \in U \times \mathbb{R} \mid (x, y) \in U\}$, as a submanifold of $U \times \mathbb{R}$. We denote by $\langle\cdot,\cdot\rangle$ the standard scalar product on $\mathbb{R}^3$.

(a) Show that $(M, \langle\cdot,\cdot\rangle|_{TM\times TM})$ and, hence, $(U, g = (\mathrm{Id_U} \times \mathrm{F})^*\langle\cdot,\cdot\rangle|_{TM\times TM})$ is a Riemannian manifold.

(b) Show that the sectional curvature $K(\Pi)$ of $(U, g)$ depends only on the base point $p$ of the tangent plane $\Pi \subset T_pU$, i.e. $K \in C^\infty(U)$.

(c) For a normal field $\xi : U \to T\mathbb{R}^3$, $p \mapsto \xi_p \in (T_{(p,F(p))}M)^\perp$, of your choice describe the Weingarten map $S^\xi$ of $(U, g)$ explicitly.

(d) Find and prove a formula for $K$ in terms of $F$ and its first and second partial derivatives.

(e) Show that $K(p) > 0$ $\forall p \in U$ if and only if $F$ is either a strictly convex map or a strictly concave map. $F$ is called strictly convex (concave) if its Hessian $H^F$ with respect to the standard flat connection of $U \subset \mathbb{R}^2$ is positive definite (negative definite) at each $p \in U$.

**7. (12 (2+2+2+6) points)**
Let $I \subset \mathbb{R}$ be an open interval. For a smooth curve $\gamma : I \to \mathbb{R}^3$ we define the arc-length of $\gamma$ from $\gamma(t_0)$ to $\gamma(t)$ to be

$$L(t) = \int_{t_0}^{t} \sqrt{\langle\dot\gamma(s), \dot\gamma(s)\rangle}ds,\ t_0, t \in I,\ t \geq t_0.$$

(a) Let $\gamma(t) = (r\cos t, r\sin t, t)$, $r > 0$. Calculate $L(t)$ for $t_0$ fixed.

(b) For $a > 0$, $b < 0$ consider $\gamma(t) = (ae^{bt}\cos t, ae^{bt}\sin t, 1)$. Show that for any $t_0 \in \mathbb{R}$ the limit $\lim_{t\to\infty} L(t)$ is finite.

(c) Under the additional assumption that $\gamma : I \to \mathbb{R}^3$ is an embedding, show that the second fundamental form of the Riemannian submanifold $\gamma(I) \subset (\mathbb{R}^3, \langle \cdot, \cdot \rangle)$ is given by

$$II(\dot{\gamma}, \dot{\gamma}) = \ddot{\gamma} - \frac{\partial \ln\left(\frac{\partial L}{\partial t}\right)}{\partial t} \dot{\gamma}.$$

(d) As above, assume that $\gamma = (\gamma^1, \gamma^2, \gamma^3)^T$ is an embedding. Furthermore assume that $\dot{\gamma}_1(t) \neq 0 \; \forall t \in I$. Show that the two vector fields $X_\gamma, Y_\gamma$ along $\gamma(I)$,

$$X_\gamma = (-\dot{\gamma}_2, \dot{\gamma}_1, 0)^T, \; Y_\gamma = (-\dot{\gamma}_3, 0, \dot{\gamma}_1)^T, \; X, Y \in \Gamma_\gamma(T\mathbb{R}^3),$$

are orthogonal to $\dot{\gamma}$. Calculate $\overline{\nabla}^{\mathrm{nor}}_{\dot{\gamma}} X_\gamma$ and $\overline{\nabla}^{\mathrm{nor}}_{\dot{\gamma}} Y_\gamma$ ($\overline{\nabla}$ denotes the flat connection in $\mathbb{R}^3$).

*(Hints: If you have trouble with the tangential and normal parts in the calculations of (c) and (d), recall how to project a vector $v$ in the $n$-dimensional Euclidean space $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ orthogonally onto the line of another non-zero vector $w$: $\mathrm{pr}_{\mathbb{R}w}(v) = \frac{\langle v, w \rangle}{\langle w, w \rangle} w$.)*

**8. (4 points)**
Let $(M, g)$ be a 2-dimensional isometrically embedded Riemannian submanifold of $(\mathbb{R}^3, \langle \cdot, \cdot \rangle)$, such that the image of $M$ under the embedding is oriented. Since $\dim M = 2$, the sectional curvature $K$ can be viewed as a function on $M$, i.e. $K \in C^\infty(M)$. For an isometric embedding $F : M \to \mathbb{R}^3$, let $\xi : F(M) \to T\mathbb{R}^3$ be a unit vector field on $F(M)$, such that $\xi_q \perp T_q F(M)$ for all $q \in F(M)$. The principal curvatures $k_1, k_2 \in C(F(M))$ of the the surface $F(M) \subset \mathbb{R}^3$ are defined as the eigenvalues of the corresponding shape tensor $S^\xi$. Show that $k_1(F(p))k_2(F(p)) = K(p)$ for all $p \in M$. Conclude that the product of the principal curvatures does neither depend on the chosen unit normal field $\xi$ nor on the chosen isometric embedding $F$.

**9. (8 points)**
Let $f \in C^\infty(\mathbb{R}^3)$, such that $c \in \mathbb{R}$ is a regular value of $f$. Consider $M := f^{-1}(c)$ as a Riemannian submanifold of $(\mathbb{R}^3, \langle \cdot, \cdot \rangle)$.

(a) Find and prove formulae in terms of $f$ and its derivatives for the second fundamental form, the shape tensor, and the sectional curvature of $(M, \langle \cdot, \cdot \rangle|_{TM \times TM})$ with respect to the unit normal field $\xi = \frac{\mathrm{grad}(f)}{\sqrt{\langle \mathrm{grad}(f), \mathrm{grad}(f) \rangle}}$ along $M$.

(b) For $A, B, C > 0$, let $f = Ax^2 + By^2 + Cz^2$, $c = 1$, and define $M$ as in (a). Calculate the sectional curvature $K$ of $M$.