Universität Hamburg

Department Mathematik
Analysis und Differentialgeometrie

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Differential geometry **summer term 2020**

Exercise sheet 10

— submission deadline: 10.7.2020 —

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1. (8 points)
Let $F : M \to N$ be an isometry of pseudo-Riemannian manifolds (M, g) and (N, h).

a) Show that F maps geodesics of (M, g) w.r.t. its Levi-Civita connection to geodesics of (N, h) w.r.t. its Levi-Civita connection.

b) Show that the exponential map is natural in the sense that $F \circ \exp_p = \exp_{F(p)} \circ dF_p$ whenever defined for all $p \in M$.

2. (8 points)
We endow the submanifold $S^2 \subset \mathbb{R}^3$ with a Riemannian metric g the following way. For $v, w \in T_pS^2 \subset T_p\mathbb{R}^3$, we define $g_p(v, w) = \langle v, w\rangle$, where $\langle \cdot, \cdot\rangle$ denotes the Euclidean scalar product in $\mathbb{R}^3$. Show that the geodesics in (S^2, g) are curves contained the great circles of S^2 with constant velocity.
(Hint: If you have trouble with the calculations, determine for your favourite chart (U, φ) of S^2 the pullback F^*g of g to $\varphi(U) \subset \mathbb{R}^2$, where $F = \iota \circ \varphi^{-1}$, ι denoting the immersion $\iota : S^2 \to \mathbb{R}^3$. Note that $F = (x \circ F, y \circ F, z \circ F)$, where (x, y, z) are the canonical coordinates in $\mathbb{R}^3$. So you only need to calculate the differential of the components of F, that is $d(x \circ F)$, $d(y \circ F)$, and $d(z \circ F)$ and plug the results into $\langle \cdot, \cdot\rangle = dx^2 + dy^2 + dz^2$ to obtain F^*g.)

3. (Bonus: 4 points)
Describe the exponential map from Analysis I, $x \mapsto e^x$, as an exponential map of a connection on a fitting smooth manifold.