1. (8 points)
Let $F : M \rightarrow N$ be an isometry of pseudo-Riemannian manifolds $(M, g)$ and $(N, h)$.

a) Show that $F$ maps geodesics of $(M, g)$ w.r.t. its Levi-Civita connection to geodesics of $(N, h)$ w.r.t. its Levi-Civita connection.

b) Show that the exponential map is natural in the sense that $F \circ \exp_p = \exp_{F(p)} \circ dF_p$ whenever defined for all $p \in M$.

2. (8 points)
We endow the submanifold $S^2 \subset \mathbb{R}^3$ with a Riemannian metric $g$ the following way. For $v, w \in T_p S^2 \subset T_p \mathbb{R}^3$, we define $g_p(v, w) = \langle v, w \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product in $\mathbb{R}^3$. Show that the geodesics in $(S^2, g)$ are curves contained the great circles of $S^2$ with constant velocity.

(Hint: If you have trouble with the calculations, determine for your favourite chart $(U, \varphi)$ of $S^2$ the pullback $F^* g$ of $g$ to $\varphi(U) \subset \mathbb{R}^2$, where $F = \iota \circ \varphi^{-1}$, $\iota$ denoting the immersion $\iota : S^2 \rightarrow \mathbb{R}^3$. Note that $F = (x \circ F, y \circ F, z \circ F)$, where $(x, y, z)$ are the canonical coordinates in $\mathbb{R}^3$. So you only need to calculate the differential of the components of $F$, that is $d(x \circ F)$, $d(y \circ F)$, and $d(z \circ F)$ and plug the results into $\langle \cdot, \cdot \rangle = dx^2 + dy^2 + dz^2$ to obtain $F^* g$.)

3. (Bonus: 4 points)
Describe the exponential map from Analysis I, $x \mapsto e^x$, as an exponential map of a connection on a fitting smooth manifold.