Differential geometry
Lecture 9: Infinitesimal generators of one parameter groups of diffeomorphisms and the Lie derivative of vector fields

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1. Infinitesimal generators of one parameter groups of diffeomorphisms

2. The Lie derivative of vector fields

Recap of lecture 8:

- defined the **differential of smooth maps** as a smooth map between tangent bundles
- defined **Lie algebras** & **Lie bracket of vector fields**
- showed that partial derivatives **commute**
- defined **integral curves** and **(local) flows** of vector fields
- defined **(local) one parameter groups of diffeomorphisms**
- showed that (local) flows of vector fields **are** (local) one parameter groups of diffeomorphisms

We have seen that local flows are local one parameter groups of diffeomorphisms.

Question: Is there a meaningful way to reverse the direction in that statement?

Answer: Yes!

Definition

Let $\varphi : I \times U \to M$ be a local one parameter group of diffeomorphisms. The **infinitesimal generator of** φ is defined to be the map

$$U \ni p \mapsto \left.\frac{\partial}{\partial t}\right|_{t=0} (\varphi_t(p)) \in T_pM.$$

$\rightsquigarrow$ does the above equation define a local vector field?

Lemma

Infinitesimal generators of local one parameter group of diffeomorphisms $\varphi : I \times U \to M$ **are** local vector fields in $\mathfrak{X}(U)$. Infinitesimal generators of one parameter groups of diffeomorphisms $\varphi : \mathbb{R} \times M \to M$ are **complete**.

Proof: (next page)

(continuation of proof)

- any local one parameter group of diffeomorphisms is **smooth** as a map
- hence the map $X : p \mapsto X_p := \frac{\partial}{\partial t}\Big|_{t=0} (\varphi_t(p))$ is **smooth**, meaning that $X \in \mathfrak{X}(U)$
- in particular for any **global** one parameter group of diffeomorphisms $\varphi : \mathbb{R} \times M \to M$, $X \in \mathfrak{X}(M)$
- integral curves at $p \in M$ of X are given by

$$t \mapsto \varphi_t(p)$$

  and are defined **for all** $t \in \mathbb{R}$
- thus: X is **complete**. □

Recall that locally, smooth m-dim. submanifolds of $\mathbb{R}^n$ can be written as the **graph** of a smooth map $f : U \subset \mathbb{R}^m \to \mathbb{R}^{n-m}$.
$\rightsquigarrow$ (id_{TU}, df) "**pushes**" a vector field from the domain of definition of f to its graph, and on the other hand we know how to "**pull back**" vector fields in $U \times \mathbb{R}^{n-m}$ that are tangential to $\mathrm{graph}(f)$ to U.
Question: How can we generalize these concepts in a **coordinate free manner** for smooth manifolds?
Answer: (next page)

Definition

Let $F : M \to N$ be a diffeomorphism and let $X \in \mathfrak{X}(M)$, $Y \in \mathfrak{X}(N)$. The **pushforward** of X under F is the vector field $F_*X \in \mathfrak{X}(N)$ given by

$$(F_*X)_q := dF_{F^{-1}(q)}\left(X_{F^{-1}(q)}\right) \quad \forall q \in N.$$

The **pullback** of Y under F is the vector field $F^*Y \in \mathfrak{X}(M)$ given by

$$(F^*Y)_p := d(F^{-1})_{F(p)}\left(Y_{F(p)}\right) \quad \forall p \in M.$$

Note that $d(F^{-1})_{F(p)} = (dF_p)^{-1}$ for all $p \in M$.

Note: Diffeomorphisms map integral curves γ of vector fields X to integral curves $F \circ \gamma$ of F_*X. **[**Exercise!**]**

Assume that $X \in \mathfrak{X}(M)$ does not vanish everywhere. Then near **any** point where X does not vanish we can find local coordinates on M in which X has a particularly **simple form**:

Proposition A

Let $X \in \mathfrak{X}(M)$ and $p \in M$, such that $X_p \neq 0$. Then there exist local coordinates on an open neighbourhood $U \subset M$ of p, such that X is of the form

$$X_q = \left.\frac{\partial}{\partial x^1}\right|_q$$

for all $q \in U$.

Proof:

- X is a section in TM, hence in particular **continuous**
- find **open neighbourhood** U of $p \in M$ such that $X_q \neq 0$ $\forall\, q \in U$
- w.l.o.g.: U is contained in a **chart neighbourhood** of an atlas on M (continued on next page)

(continuation of proof)

- choose a local coordinate system $\phi = (y^1, \ldots, y^n)$ on U and let $(u^1, \ldots, u^n)$ denote the canonical coordinates on $\mathbb{R}^n$
- after possibly **shrinking** U and **re-ordering** the y^i's, can assume w.l.o.g. that $\phi_*(X) \in \mathfrak{X}(\phi(U))$ is **transversal** along the inclusion map $\{u^1 = 0\} \cap \phi(U) \hookrightarrow \mathbb{R}^n$ to $N := \{u^1 = 0\} \cap \phi(U)$, meaning that

$$(\phi_* X)_q \notin T_q N \cong T_q\{u^1 = 0\} \subset T_q\mathbb{R}^n$$

  for all $q \in N$
- after again possibly shrinking U, let $\Phi : I \times \phi(U) \to \mathbb{R}^n$ denote a **local flow** of $\phi_* X$

(continuation of proof)

- since
$$(\phi_* X)_q = \frac{\partial}{\partial t}\bigg|_{t=0} \Phi_t(q) \neq 0$$
by the **transversality condition**, we obtain **[**recall: F local diffeo near $p \Leftrightarrow dF_p$ invertible**]** after possibly **shrinking** I that
$$F := \Phi|_{I\times N} : I \times N \to \Phi(I \times N)$$
is a diffeomorphism, where I is the "time" part so that $\Phi(t, q) := \Phi_t(q)$, and $\Phi(I \times N) \subset \mathbb{R}^n$ is open
- denote the canonical coordinates in I by u^1 and in N by $(u^2, \ldots, u^n)$, this is **compatible** with the canonical inclusion $I \times N \subset \mathbb{R}^n$
- hence:
$$dF_{(u^1,u^2,\ldots u^n)}\left(\frac{\partial}{\partial u^1}\bigg|_{(u^1,u^2,\ldots,u^n)}\right) = (\phi_* X)_{\Phi(u^1,u^2,\ldots,u^n)}$$
for all $(u^1, \ldots, u^n) \in I \times N$

(continuation of proof)

- define **coordinates** on $\phi^{-1}(\phi(U) \cap \Phi(I \times N)) \subset M$ by

  $\psi = (x^1, \ldots, x^n) :=$
  $F^{-1} \circ \phi : \phi^{-1}(\phi(U) \cap \Phi(I \times N)) \to F^{-1}(\phi(U) \cap (I \times N)) \subset \mathbb{R}^n$

- obtain for the **local formula** of X in the local coordinate system ψ and **all** $q \in \phi^{-1}(\phi(U) \cap \Phi(I \times N))$

$$X_q = \left.\frac{\partial}{\partial x^1}\right|_q$$

□

In local coordinates as the ones constructed in Proposition A, local flows look particularly simple:

Corollary A

Any local flow of X near p as in Proposition A is, if $X_p \neq 0$, in the local coordinate system $\psi = (x^1, \dots, x^n)$ of the form

$$\psi(\varphi_t(q)) = \psi(q) + te_1,$$

for all $q \in U$, where e_1 denotes the first unit vector in $\mathbb{R}^n$ in canonical coordinates, for $|t|$ small enough. Furthermore

$$d\varphi_t \left(\left. \frac{\partial}{\partial x^i} \right|_q \right) = \left. \frac{\partial}{\partial x^i} \right|_{\psi^{-1}(\psi(q)+te_1)}$$

for all $q \in U$ and t small enough, where we understand the differential of φ_t for t fixed.

Next we want to relate the Lie algebra structure on vector fields to their local flows. In order to do so, we need the following:

Definition

Let $\phi : M \to N$ be a smooth map. Two vector fields $X \in \mathfrak{X}(M)$ and $\overline{X} \in \mathfrak{X}(N)$ are called ϕ-**related** if $d\phi(X) = \overline{X}_\phi$. One then writes $X \sim_\phi \overline{X}$. Equivalent definition: $X \sim_\phi \overline{X}$ if $X(f \circ \phi) = Y(f) \circ \phi$ for all $f \in C^\infty(N)$.

ϕ-related is preserved under the Lie bracket:

Lemma A

Let $\phi : M \to N$ be smooth, $X, Y \in \mathfrak{X}(M)$ and $\overline{X}, \overline{Y} \in \mathfrak{X}(N)$, such that $X \sim_\phi \overline{X}$ and $Y \sim_\phi \overline{Y}$. Then $[X, Y] \sim_\phi [\overline{X}, \overline{Y}]$.

Proof:

$$[X, Y](f \circ \phi) = X(Y(f \circ \phi)) - Y(X(f \circ \phi))$$
$$= X(\overline{Y}(f) \circ \phi) - Y(\overline{X}(f) \circ \phi) = (\overline{X}(\overline{Y}(f)) - \overline{Y}(\overline{X}(f))) \circ \phi$$

for all $f \in C^\infty(N)$. $\square$

Corollary B

For **diffeomorphisms** $F : M \to N$,

$$F_*[X, Y] = [F_*X, F_*Y]$$

for all $X, Y \in \mathfrak{X}(M)$ and

$$F^*[\overline{X}, \overline{Y}] = [F^*\overline{X}, F^*\overline{Y}]$$

for all $\overline{X}, \overline{Y} \in \mathfrak{X}(N)$.

We further have the following fact which is essential for studying submanifolds:

Remark

In the case that ϕ is an embedding and $\dim(M) < \dim(N)$, the previous lemma implies that (**locally and globally**) $[\overline{X}, \overline{Y}] \circ \phi$ does **not** depend on the (local) **extensions** of $\overline{X} \circ \phi$ and $\overline{Y} \circ \phi$ to vector fields on in N open neighbourhoods of points in $\phi(M)$.

The Lie bracket of vector fields and their flows are related as follows:

Proposition B

Let $X, Y \in \mathfrak{X}(M)$ and for $p \in M$ arbitrary but fixed let $\varphi : I \times U \to M$ be a local flow of X near p. Then

$$[X, Y]_p = \frac{\partial}{\partial t}\bigg|_{t=0} (\varphi_t^* Y)_p.$$

Proof: (next page)

(continuation of proof)

- observe: $\frac{\partial}{\partial t}\big|_{t=0} (\varphi_t^* Y)_p$ is actually a **well-defined** expression
- follows from $(\varphi_t^* Y)_p \in T_pM$ for all $t \in I$ and the fact that T_pM is a **real vector space**
- in the following: $d\varphi_t =$ differential of φ_t for $t \in I$ **fixed**.
- first case: $X_p \neq 0$
- Prop. A & Cor. B $\rightsquigarrow$ w.l.o.g. assume that we have chosen **local coordinates** $(x^1, \dots, x^n)$ on $U \subset M$ with $p \in M$, such that $X_q = \frac{\partial}{\partial x^1}\big|_q$ for all $q \in U$
- recall: φ is a local one parameter group of diffeomorphisms, hence $\varphi_{-t} = \varphi_t^{-1}$ whenever defined
- implies that for $|t|$ small enough we have

$$(\varphi_t^* Y)_p = d(\varphi_t^{-1})_{\varphi_t(p)} \left(Y_{\varphi_t(p)}\right) = (d\varphi_{-t})_{\varphi_t(p)} \left(Y_{\varphi_t(p)}\right)$$

(continuation of proof)

- observe that Corollary A shows

$$(d\varphi_{-t})_{\varphi_t(p)} : \frac{\partial}{\partial x^i}\bigg|_{\varphi_t(p)} \mapsto \frac{\partial}{\partial x^i}\bigg|_{\psi^{-1}(\psi(\varphi_t(p))-te_1)}$$
$$= \frac{\partial}{\partial x^i}\bigg|_{\psi^{-1}(\psi(p)+te_1-te_1)} = \frac{\partial}{\partial x^i}\bigg|_p$$

- in the local coordinates $(x^1, \ldots, x^n)$, Y is of the form

$$Y_q = \sum_{i=1}^{n} Y^i(q)\, \frac{\partial}{\partial x^i}\bigg|_q$$

  for all $q \in U$

- this implies

$$(\varphi_t^* Y)_p = \sum_{i=1}^{n} Y^i(\varphi_t(p))\, \frac{\partial}{\partial x^i}\bigg|_p$$

(continuation of proof)

- hence:

$$\left.\frac{\partial}{\partial t}\right|_{t=0} (\varphi_t^* Y)_p = \sum_{i=1}^{n} dY^i \left(\left.\frac{\partial}{\partial x^1}\right|_p \right) \left.\frac{\partial}{\partial x^i}\right|_p$$

  which coincides with $[X, Y]_p = \left[\frac{\partial}{\partial x^i}, Y\right]_p$

- next case: $X_p = 0$
- if $X_q = 0$ **for all** q in an **open neighbourhood** U of p, the local flow of X restricted to U will be the **identity** for all $t \in I$
- hence,

$$\left.\frac{\partial}{\partial t}\right|_{t=0} (\varphi_t^* Y)_p = 0$$

- furthermore for any $f \in C^\infty(M)$ observe that $X(f)$ **vanishes** on U and thus

$$[X, Y]_p(f) = X_p(Y(f)) - Y_p(X(f)) = 0$$

(continuation of proof)

- last case: $X_p = 0$ and X **does not vanish identically** on some open neighbourhood of p
- let $U \subset M$ be a **compactly embedded** open neighbourhood of p and choose a sequence $\{p_n\}_{n\in\mathbb{N}}$, $\lim_{n\to\infty} p_n = p$, such that $X_{p_n} \neq 0$ and $p_n \neq p$ for all $n \in \mathbb{N}$
- then
$$[X, Y]_{p_n} = \frac{\partial}{\partial t}\bigg|_{t=0} (\varphi_t^* Y)_{p_n}$$
for all $n \in \mathbb{N}$
- **continuity in the base point** of both sides of the above expression $\rightsquigarrow$ by taking limits $n \to \infty$ on both sides conclude that $[X, Y]_p = \frac{\partial}{\partial t}\big|_{t=0} (\varphi_t^* Y)_p$ **as claimed** □

As announced, we can now define the Lie derivative of vector fields which is **one way** of measuring infinitesimal changes in vector fields in the direction of an other vector field:

Definition

The **Lie derivative** of a vector field $Y \in \mathfrak{X}(M)$ in direction of $X \in \mathfrak{X}(M)$ is defined as

$$\mathcal{L}_X(Y) := [X, Y] \in \mathfrak{X}(M).$$

Examples

- $\mathcal{L}_X(X) = 0$
- $\mathcal{L}_{\frac{\partial}{\partial x^i}}\left(\frac{\partial}{\partial x^j}\right) = 0$ for any local coordinates $(x^1, \ldots, x^n)$
- $X, Y \in \mathfrak{X}(\mathbb{R}^n)$,

$$X = \sum_i c^i \frac{\partial}{\partial u^i}, \quad Y = \sum_i Y^i \frac{\partial}{\partial u^i},$$

$c = (c^1, \ldots, c^n)$ a constant vector, then

$$[X, Y] = \sum_i dY^i(c) \frac{\partial}{\partial u^i}$$

END OF LECTURE 9

Next lecture:

- the cotangent bundle
- one forms
- tensor fields