Differential geometry
Lecture 8: Lie bracket of vector fields, integral curves, flows

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Recap of lecture 7:

- defined **tangent bundle**
- discussed **Strong Whitney Embedding Theorem**
- defined **vector fields** on smooth manifolds
- discussed action of vector fields on smooth functions, also in local coordinates
- defined **coordinate vector fields**
- showed that vector fields can be viewed as **derivations on** $C^\infty(M)$
- erratum: forgot to give example of (non-canonical) choice of VB structure on $\bigsqcup_{p \in M} T_p M$

Having defined the tangent bundle of smooth manifolds and its sections, smooth vector fields, we have all necessary tools at hand to give a **global** definition of the differential of smooth maps:

Definition

Let M, N be smooth manifolds, $F : M \to N$ a smooth map. The **differential of** F is defined as the smooth map

$$dF : TM \to TN, \quad dF|_{\pi^{-1}(p)} = dF_p \quad \forall p \in M.$$

In local coordinates $(x^1, \ldots, x^m)$ of M and $(y^1, \ldots, y^n)$ of N with appropriate domain we have

$$dF\left(\frac{\partial}{\partial x^i}\right) = \sum_{j=1}^{n} \frac{\partial F^j}{\partial x^i}\frac{\partial}{\partial y^j}, \quad F^j = y^j \circ F, \quad \forall 1 \leq i \leq m.$$

The (non-pointwise) Jacobi matrix in given local coordinates is defined similarly by allowing the basepoint to vary and, as a map from chart neighbourhoods in M to $\mathrm{Mat}(n \times n, \mathbb{R})$, is also smooth.

Next, some Lie algebra:

Definition

Let V be a real vector space. A **Lie bracket** on V is a **skew-symmetric bilinear map**

$$[\cdot,\cdot] : V \times V \to V, \quad (X, Y) \mapsto [X, Y]$$

that fulfils the **Jacobi identity**

$$[X,[Y,Z]] = [[X,Y],Z] + [Y,[X,Z]]$$

for all $X, Y, Z \in V$. A vector space V together with a Lie bracket is called **Lie algebra**.

Note: The above Jacobi identity can be written as

$$\sum_{\text{cyclic}} [X,[Y,Z]] = 0 \quad \forall X, Y, Z \in V,$$

which is easier to remember.

Examples

- $\mathbb{R}^3$ together with the cross product $[X, Y] := X \times Y$ is a Lie algebra
- $\mathrm{End}(\mathbb{R}^n) = \mathrm{Mat}(n \times n, \mathbb{R})$ with $[A, B] := AB - BA$ is a Lie algebra
- for any commutative real algebra A, $\mathrm{Der}(A)$ is a possibly infinite dimensional Lie algebra with $[D_1, D_2](a) := D_1(D_2(a)) - D_2(D_1(a))\ \forall a \in A$, $D_1, D_2 \in \mathrm{Der}(A)$

$\rightsquigarrow$ $\mathfrak{X}(M)$ is isomorphic as $C^\infty(M)$-module to $\mathrm{Der}(C^\infty(M))$, points to Lie algebra structure on the real vector space of vector fields

Note: $\dim(\mathfrak{X}(M)) = \infty$ if $\dim(M) > 0$

Proposition A

The bilinear map on vector fields on a smooth manifold M

$$[\cdot,\cdot] : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M), \quad (X, Y) \mapsto [X, Y],$$
$$[X, Y](f) := X(Y(f)) - Y(X(f))$$

$\forall X, Y \in \mathfrak{X}(M)\ \forall f \in C^\infty(M)$, is a **Lie bracket on the vector space** $\mathfrak{X}(M)$.

Proof:

- we have $X(f) \in C^\infty(M)$, $Y(f) \in C^\infty(M)$, hence $X(Y(f)) - Y(X(f)) \in C^\infty(M)$ for all $f \in C^\infty(M)$.
- at each $p \in M$ we have

$$[X, Y]_p(f) = X_p(Y(f)) - Y_p(X(f)) \quad \textbf{[}\text{note: skew in } X, Y\textbf{]}$$

- hence, $[X, Y]_p : C^\infty(M) \to \mathbb{R}$ is $\mathbb{R}$-linear
- Leibniz rule: stubbornly write down $[X, Y]_p(fg)$ for $f, g \in C^\infty(M)$ arbitrary ✓
- summarizing: $[X, Y]$ **is** a vector field $\forall X, Y \in \mathfrak{X}(M)$ and $[\cdot,\cdot]$ is indeed a Lie bracket □

Recall that partial derivatives $\frac{\partial}{\partial u^i}$ in $\mathbb{R}^n$ commute. **Question:** Is the same true for general smooth manifolds?
Answer: Yes!

Lemma

Let M be a smooth manifold and $\varphi = (x^1, \dots, x^n)$ be a local coordinate system. Then

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0$$

$\forall\ 1 \leq i \leq n$, $1 \leq j \leq n$., i.e. coordinate vector fields **commute**.

Remark: The above lemma can be formulated as

$$\frac{\partial}{\partial x^i}\left(\frac{\partial f}{\partial x^j}\right) = \frac{\partial}{\partial x^j}\left(\frac{\partial f}{\partial x^i}\right)$$

for all $f \in C^\infty(M)$.
Notation: $\frac{\partial^2 f}{\partial x^i \partial x^j} := \frac{\partial}{\partial x^i}\left(\frac{\partial f}{\partial x^j}\right)$, analogously for higher derivatives.
Warning: Careful when changing coordinates!

Proof (of the lemma):

- let $\widehat{f}$ be a coordinate representative for f in local coordinates $(x^1, \ldots, x^n)$, so that $\widehat{f} = f \circ \varphi^{-1}$
- $\rightsquigarrow f = \widehat{f}(x^1, \ldots, x^n)$
- Q: Do the coordinate vector fields act on $\widehat{f}(x^1, \ldots, x^n)$ as expected?
- A: Yes! Obtain

$$\left(\frac{\partial}{\partial x^i}(f)\right)(p) = \left(\frac{\partial}{\partial u^i}\left(\widehat{f}(u^1, \ldots, u^n)\right)\right)(\varphi(p)).$$

- hence, $\frac{\partial}{\partial x^i}(f) = \frac{\partial}{\partial u^i}\left(\widehat{f}\right) \circ \varphi$

  [note: actually follows from Lecture 7, i.e. $\widehat{\frac{\partial}{\partial x^i}} = \frac{\partial}{\partial u^i}$]
- summarizing:

$$\frac{\partial}{\partial x^i}\left(\frac{\partial}{\partial x^j}(f)\right) = \frac{\partial}{\partial x^i}\left(\frac{\partial}{\partial u^j}\left(\widehat{f}\right) \circ \varphi\right) = \frac{\partial}{\partial u^i}\left(\frac{\partial}{\partial u^j}\left(\widehat{f}\right)\right) \circ \varphi$$

$$= \frac{\partial}{\partial u^j}\left(\frac{\partial}{\partial u^i}\left(\widehat{f}\right)\right) \circ \varphi = \ldots = \frac{\partial}{\partial x^j}\left(\frac{\partial}{\partial x^i}(f)\right) \qquad \square$$

Question: Is the Lie bracket on vector fields $C^\infty(M)$-linear?
Answer: No!

Lemma

For any smooth manifold M

$$[X, fY] = df(X)Y + f[X, Y] \quad \forall X, Y \in \mathfrak{X}(M),\ \forall f \in C^\infty(M).$$

Proof: Exercise!

Smooth maps $F : U \subset \mathbb{R}^m \to \mathbb{R}^n$, $U \subset \mathbb{R}^m$ open, can be visualized as their respective graphs. We see that for each vector field $X \in \mathfrak{X}(U)$ the smooth map

$$U \ni p \mapsto (X_p, dF_p(X_p)) \in T_{(p,F(p))}(U \times \mathbb{R}^n)$$

describes not a vector field in $U \times \mathbb{R}^n$, but a vector field **along** the graph of F that is **tangential to** the graph of F. If we only look at the second factor **[**note: first factor is the identity, no new info**]** we might call $p \mapsto dF_p(X_p) \in T_{F(p)}\mathbb{R}^n$ a **vector field along** F.
$\rightsquigarrow$ can be easily generalized for smooth manifolds (next page)

Definition

Let $\phi : M \to N$ be a smooth map and let $X \in \mathfrak{X}(M)$. The smooth map

$$M \ni p \mapsto (d\phi(X))_p = d\phi_p(X_p) \in T_{\phi(p)}N$$

is called a **vector field along** ϕ.

Probably the most prominent examples of vector fields along maps are **velocity vector fields** of smooth curves:

Definition

Let $I \subset \mathbb{R}$ be an interval (equipped with canonical coordinate t), M a smooth manifold, and $\gamma : I \to M$ a smooth curve. The **velocity vector field** (or simply **velocity**) of γ is the vector field along γ

$$\gamma' := d\gamma\left(\frac{\partial}{\partial t}\right), \quad t \mapsto \gamma'(t).$$

(continued on next page)

Definition

(continuation)
Note that the **explicit form** of $\gamma'(t)$ depends on the local coordinates $\varphi = (x^1, \dots, x^n)$ on M:

$$\gamma'(t) = \sum_{i=1}^{n} \frac{\partial \gamma^i}{\partial t}(t) \left. \frac{\partial}{\partial x^i} \right|_{\gamma(t)} \in T_{\gamma(t)}M$$

for all $t \in I$, where $\gamma^i = x^i(\gamma)$ for all $1 \le i \le n$.

Remark: The above relates our definition of tangent vectors in $\mathbb{R}^n$ as equivalence classes of smooth curves to smooth manifolds.

Now, recall the definition of **integral curves** in setting of analysis on $\mathbb{R}^n$.
Question: Can this be generalized to smooth manifolds?
Answer: Of course!

Definition

Let $X \in \mathfrak{X}(M)$ for a smooth manifold M. An **integral curve** of X at $p \in M$ is a smooth curve $\gamma : I \to M$, where $I \subset \mathbb{R}$ is an interval, $0 \in I$, such that $\gamma(0) = p$ and

$$\gamma'(t) = X_{\gamma(t)}$$

for all $t \in I$. An integral curve $\gamma : I \to M$ of X is called **maximal** if there is no interval $\widetilde{I} \supset I$, such that $\widetilde{I} \setminus I \neq \emptyset$ and there exists an integral curve $\widetilde{\gamma} : \widetilde{I} \to M$ of X with $\widetilde{\gamma}|_I = \gamma$. A vector field X is called **complete** if every maximal integral curve $\gamma : I \to M$ is defined on $I = \mathbb{R}$.

Questions:

- What **kind** of equation is $\gamma'(t) = X_{\gamma(t)}$?
- How does it look like **locally**?
- Are **local solutions** w.r.t. some initial values unique?
- Are **maximal solutions** w.r.t. some initial values unique?
- Do local solutions depend **smoothly** on initial values?

Recommended reading: [A] "*Ordinary Differential Equations*" (3rd edition, 1984), V.I. Arnold, Springer Universitext

Answers:

- $\gamma'(t) = X_{\gamma(t)}$ is a first order ODE.
- Like a first order ODE for a curve in $\mathbb{R}^n$. For a local coordinate system $\varphi = (x^1, \ldots, x^n)$ on M so that the domain of φ contains $\gamma(0)$, the equation can be written in has coordinate representation

$$\dot{u}^i = \widehat{X}^i(u^1, \ldots, u^n), \; u^i(0) = x^i(p) \quad \forall 1 \leq i \leq n.$$

- Yes, local solutions coincide if defined on same interval [A, Ch. 2.7]
- Yes [A, Ch. 2.7].
- Yes [A, Ch. 2.7].

Examples

- Let $X \in \mathfrak{X}(\mathbb{R}^2)$ in canonical coordinates $(u^1, u^2) = (x, y)$ be given by
$$X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$
Its integral curves at any point $(x_0, y_0) \in \mathbb{R}^2$ are of the form
$$\gamma : t \mapsto \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$
- For $X \in \mathfrak{X}(\mathbb{R}^n)$, $X = \sum c^i \frac{\partial}{\partial u^i}$, c^i constant for all $1 \leq i \leq n$, its maximal integral curves are of the form
$$\gamma : t \mapsto p + (tc^1, \ldots, tc^n).$$

In general, it is **very difficult** to answer whether a given vector field **is complete or not**. We have, however, the following results: (next page)

Proposition B

1. Vector fields on **compact smooth manifolds** are complete.
2. Vector fields with **compact support** are complete.

Proof (sketch):

- **1)** assume $\gamma : (a, b) \to M$ maximal integral curve of fixed VF $X \in \mathfrak{X}(M)$ and $b < \infty$
- show that $\lim_{t \to b} \gamma(t)$ converges to a point $q \in M$
- find integral curve $\widetilde{\gamma}$ at q of X
- show that $\widetilde{\gamma}$ **extends** γ, get contradiction to $b < \infty$
- same if $a > -\infty$
- **2)** assume $\gamma : (a, b) \to M$ maximal integral curve of fixed VF $X \in \mathfrak{X}(M)$ and $b < \infty$
- proceed as before, except need to additionally show that $\lim_{t \to b} \gamma(t)$ is in $\mathrm{supp}(X)$
- note: every maximal integral curve at $p \in \overline{M \setminus \mathrm{supp}(X)}$ is a **constant curve** □

Question: Can we (locally) describe all integral curves of a vector fields "at once"?
Answer: Yes, leads to the following definition:

Definition

A **local one parameter group of diffeomorphisms** on a smooth manifold M is a smooth map

$$\varphi : I \times U \to M, \quad (t, p) \mapsto \varphi_t(p),$$

such that $I \subset \mathbb{R}$ is an interval containing $0 \in \mathbb{R}$, $U \subset M$ is open, $\varphi_0 = \mathrm{id}_U$, $\varphi_t : M \to M$ is a diffeomorphism for all $t \in I$, and

$$\varphi_{s+t}(p) = \varphi_s(\varphi_t(p))$$

for all $p \in U$ and all $s, t \in I$ with $(s + t) \in I$ and $\varphi_t(p) \in U$. A **one parameter group of diffeomorphisms** is a local one parameter group of diffeomorphisms with $I = \mathbb{R}$ and $U = M$.

$\rightsquigarrow$ how is this connected to integral curves of vector fields?

Definition

A **local flow** of a vector field $X \in \mathfrak{X}(M)$ is a smooth map

$$\varphi : I \times U \to M, \quad (t, p) \mapsto \varphi_t(p),$$

for some interval $I \subset \mathbb{R}$ containing $0 \in \mathbb{R}$ and an open set $U \subset M$, such that $\varphi_0 = \mathrm{id}_U$ and for every $p \in U$ fixed, the smooth curve

$$t \mapsto \varphi_t(p)$$

is an integral curve of X. This just means that

$$\frac{\partial}{\partial t}(\varphi_t(p)) = X_{\varphi_t(p)}.$$

We say that a local flow of X is defined near a point $p \in M$ if $p \in U$. A local flow of X is called **(global) flow** of X if $I = \mathbb{R}$ and $U = M$.

Question: Does every vector field admit a local flow near every point?
Answer: Yes!

Lemma

Every vector field on M admits a local flow near any given point $p \in M$.

Proof:

- fix $X \in \mathfrak{X}(M)$ and $p \in M$
- choose bump function $b : M \to \mathbb{R}$, such that on some open neighbourhood $U \subset M$ of p, $b|_U \equiv 1$
- the maximal integral curves at p of bX are each defined on $\mathbb{R}$ by Proposition B and depend smoothly on initial condition $p \in M$
- already shows: vector fields with compact support admit a global flow φ
- fix $\varepsilon > 0$ and choose an open subset $V \subset U$, such that V is an open neighbourhood of p and for all $q \in V$ and all $t \in (-\varepsilon, \varepsilon)$, $\varphi_t(q) \in U$

(continued on next page)

(continuation of proof)

- geometrically: the set V is not moved out of U by the flow of bX for $|t| < \varepsilon$
- X and bX coincide on U, hence their integral curves at all $q \in V$ for $I = (-\varepsilon, \varepsilon)$ also coincide
- hence, the flow φ of bX restricted to $(-\varepsilon, \varepsilon) \times V$ is a local flow of X □

Now we know that local flows of vector fields always exist near any given point. Furthermore, the following holds true:

Proposition C

Local flows of vector fields are local one parameter groups of diffeomorphisms.

Proof:
(next page)

(continuation of proof)

- suffices to show that for a given vector field X with two integral curves $\gamma : (a, b) \to M$ at $p = \gamma(0)$ and $\widetilde{\gamma} : (\widetilde{a}, \widetilde{b}) \to M$ with $\gamma(s) = \widetilde{\gamma}(0)$ for some $s \in (a, b)$ we have

$$\gamma(s + t) = \widetilde{\gamma}(t)$$

  for all t, such that $(s + t) \in (a, b)$ and $t \in (\widetilde{a}, \widetilde{b})$
- read: " $\widetilde{\gamma}$ **extends** γ "
- follows from the fact that $t \mapsto \gamma(s + t)$ is an integral curve of X (for s **small enough**) and uniqueness of local solutions:

$$(\gamma(s + \cdot))'(t) \overset{\text{chain rule}}{=} \gamma'(s + t) = X_{\gamma(s+t)} = X_{(\gamma(s+\cdot))(t)}.$$

- hence, for ϕ a local flow of X we obtain

$$\phi_t(\phi_s(p)) = \phi_t(\gamma(s)) = \widetilde{\gamma}(t) = \gamma(s + t) = \phi_{s+t}(p)$$

□

Immediate consequence:

Corollary

Assume that $X \in \mathfrak{X}(M)$ is **complete**. Then its flow is a **one parameter group of diffeomorphisms**.

END OF LECTURE 8

Next lecture:

- reverse direction of Proposition C
- pullbacks, pushforwards, rectifications
- Lie derivative of vector fields
- (maybe start with cotangent bundle)