Differential geometry
Lecture 7: The tangent bundle

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12. May 2020

1 The tangent bundle of a smooth manifold

2 Vector fields

Recap of lecture 6:

- recalled definition of **vector fields on** $\mathbb{R}^n$, discussed examples and action on smooth functions
- defined **vector bundles** over smooth manifolds
- defined **(local) sections** in vector bundles
- studied **transition functions** for **local trivializations** of vector bundles
- explained how to construct a vector bundle from given **vector spaces at each point** and **transition functions**
- defined term **trivializable** for vector bundles

$\rightsquigarrow$ similarly to proposition from last lecture about how to "glue" pointwise vector spaces E_p, $p \in M$, define:

Definition

Let M be an n-dimensional smooth manifold. The **tangent bundle**

$$TM := \bigsqcup_{p \in M} T_p M \to M$$

of M with projection $\pi(v) = p$ for all $v \in T_p M$ is a vector bundle of rank n.

Problem: To show that $TM \to M$ in fact **is** a vector bundle, need to define local trivializations and study **matrix part** of transition functions.

Question: How do we get this data?

Answer: Use atlas and its local coordinates on M!

Proposition A

The tangent bundle TM of any given manifold is, in fact, a vector bundle of rank n.
[Warning: There are **choices** involved!**]**

Proof:

- first, define candidates for **charts on the total space**
- choose **countable atlas** $\mathcal{A} = \{(\varphi_i = (x_i^1, \ldots, x_i^n), U_i) \mid i \in A\}$ on M
- π **smooth** by assumption $\Rightarrow \{\pi^{-1}(U_i) \mid i \in A\}$ are **open covering** of TM
- define **candidates for charts** as maps between sets

$$\psi_i : \pi^{-1}(U_i) \to \varphi_i(U_i) \times \mathbb{R}^n,$$
$$\psi_i : v \mapsto (\varphi_i(\pi(v)), v(x_i^1), \ldots, v(x_i^n))$$

- observe: each ψ_i is a **bijection**

- define **basis of topology** on TM as

$$\{\psi_i^{-1}(V) \mid i \in A,\ V \subset \varphi_i(U_i) \times \mathbb{R}^n \text{ open}\}$$

- in this topology, all ψ_i are homeomorphisms [note: this the **coarsest** possible topology on TM so that this holds]
- M & $\mathbb{R}^n$ Hausdorff & second countable, $\mathcal{A}$ countable $\Rightarrow$ TM with this topology also **Hausdorff** & **second countable**
- **transition functions** of the maps ψ_i, $i \in A$, given by

$$\psi_i \circ \psi_j^{-1} : \varphi_j(U_i \cap U_j) \times \mathbb{R}^n \to \varphi_i(U_i \cap U_j) \times \mathbb{R}^n,$$
$$(u, w) \mapsto ((\varphi_i \circ \varphi_j^{-1})(u), d(\varphi_i \circ \varphi_j^{-1})_u(w))$$

- $\rightsquigarrow$ above transition functions are **smooth**, hence ψ_i, $i \in A$, form an $2n$-**dimensional smooth atlas** on total space TM

- corresponding candidates for **local trivializations**

$$(\varphi_i^{-1} \times \mathrm{id}_{\mathbb{R}^n}) \circ \psi_i$$

  of TM are of the form

$$\begin{aligned}\tau_{ij} :&= (\varphi_i^{-1} \times \mathrm{id}_{\mathbb{R}^n}) \circ \psi_i \circ \left((\varphi_j^{-1} \times \mathrm{id}_{\mathbb{R}^n}) \circ \psi_j\right)^{-1} \\ &= (\mathrm{id}_M, d(\varphi_i \circ \varphi_j^{-1}))\end{aligned}$$

- all $\tau_{ij} : (U_i \cap U_j) \times \mathbb{R}^n \to (U_i \cap U_j) \times \mathbb{R}^n$ are smooth and of the form $(p, v) \mapsto A(p)v$, $A : U_i \cap U_j \to \mathrm{GL}(n)$
- hence: TM with the so-defined local trivializations **is** a vector bundle as claimed □

Remark: τ_{ij} and local trivializations **compatible** in sense of proposition about constructing vector bundles from local bijections and given smooth transition functions, see Lecture 6.

$\rightsquigarrow$ still abstract description of TM, locally we have a simpler picture: (next page)

Remark

Let $\varphi = (x^1, \dots, x^n)$ be a local coordinate system on M covering $p \in M$. Let $v \in T_pM$,

$$v = \sum_{i=1}^{n} v^i \left.\frac{\partial}{\partial x^i}\right|_p,$$

be arbitrary. For

$$\psi = (\varphi \circ \pi, d\varphi)$$

as in the previous Proposition A we get

$$\psi(v) = (x^1(p), \dots, x^n(p), v^1, \dots, v^n),$$

meaning that the **vector part** of $\psi(v)$ consists of the **prefactors of** v in the basis $\left\{ \left.\frac{\partial}{\partial x^i}\right|_p,\ 1 \leq i \leq n \right\}$.

Example

Let $M \subset \mathbb{R}^n$ be a smooth submanifold of codimension $0 < k < n$. locally, write M as **graph** of a smooth map

$$g : U \subset \mathbb{R}^{n-k} \to \mathbb{R}^k.$$

Pointwise, T_pM is the set of tangent vectors $v \in T_p\mathbb{R}^n$ that are in the image of dg_p. The **induced chart** on M is given by

$$\varphi : \mathrm{graph}(g) \to \mathbb{R}^{n-k}, \quad (p, g(p)) \mapsto p,$$

and the inverse of the **induced local trivialization** ψ of TM is given by

$$\psi^{-1} : U \times \mathbb{R}^{n-k} \to TM|_U, \quad (p, v) \mapsto ((p, g(p)), (v, dg_p(v))).$$

Question: Can we imagine the tangent bundle, locally, of **any** smooth manifold as in the above example?
Answer: Yes! (next page)

Theorem (Strong Whitney Embedding Theorem)

Every smooth manifold of dimension n can be realized as an **embedded submanifold** of $\mathbb{R}^{2n}$.

Proof: See Theorem 6.19 in "*Introduction to Smooth Manifolds*" (John M. Lee), Springer GTM 218.

Recall that TM is called **trivializable** if it is isomorphic as a vector bundle to $M \times \mathbb{R}^n$, $n = \dim(M)$.

Lemma

TS^1, TS^3, and TS^7 are trivializable.

Proof: S^1:

- view $S^1 = \{z\bar{z} = 1\} \subset \mathbb{C} \cong \mathbb{R}^2$
- for $z \in S^1$ fixed, iz spans T_zS^1
- define vector bundle isomorphism

$$F : S^1 \times \mathbb{R} \to TS^1, \quad (z, t) \mapsto (z, tiz)$$

- hence: total space of $TS^1 \to S^1$ is diffeomorphic to the cylinder
- S^3 & S^7: Exercise! [hint: view S^3 as the **unit quaternions** and S^7 as the **unit octonions**] □

Now we have all tools at hand to define vector fields on smooth manifolds.

Definition

Sections in the tangent bundle of a smooth manifold, $\Gamma(TM)$, are called **vector fields**. For $X \in \Gamma(TM)$ we will denote the value of X at $p \in M$ by X_p. For $U \subset M$ open, we will call elements of $\Gamma(TM|_U)$ **local vector fields**, or simply vector fields if the setting does not explicitly use the locality property. We will use the notations

$$\mathfrak{X}(M) := \Gamma(TM)$$

and

$$\mathfrak{X}(U) := \Gamma(TM|_U)$$

for $U \subset M$ open.

Remark

For a smooth manifold M and $U \subset M$ open, the two vector spaces T_pU and T_pM are canonically isomorphic via restriction of charts for all $p \in U$.

If so-defined vector fields look a bit alien at first glance, it helps to look at them through the lens of local coordinates:

Remark

In local coordinates $(\varphi = (x^1, \dots, x^n), U)$ on a smooth manifold M with induced coordinates $\psi = (\varphi \circ \pi, d\varphi)$ on $TM|_U \subset TM$, (local) vector fields X have coordinate representations of the form

$$\widehat{X} : \varphi(U) \to U \times \mathbb{R}^n, \quad \widehat{X} = (\mathrm{id}_U, d\varphi(X_{\varphi^{-1}})),$$

$\widehat{X}$ smooth. This means that locally up to the explicit notation of the base point, vector fields on smooth manifolds **are** vector fields on an open subset of $\mathbb{R}^n$.

Remark

Vector fields, similar to tangent vectors, act on $C^\infty(M)$ by

$$X(f)(p) := X_p(f) = df(X_p).$$

Thus we may write $X(f) = df(X) \in C^\infty(M)$. On the other hand, a map of the form $X : M \to TM$, $p \mapsto X_p \in T_pM$, is a vector field if $X(f) : p \mapsto df_p(X_p)$ is smooth for all $f \in C^\infty(M)$.

The above is **compatible** with writing X and f in coordinate representations:

Lemma

The coordinate representation of $X(f)$ is the same as the action of the vector part of the coordinate representation of X on the coordinate representation of f.

Proof: (next page)

(continuation of proof)
Write $\widehat{f} = f \circ \varphi^{-1}$, vector part of $\widehat{X} = d\varphi(X_{\varphi^{-1}})$. We obtain

$$X(f) \circ \varphi^{-1} = df(X) \circ \varphi^{-1}$$
$$= (d(f \circ \varphi^{-1}) \circ d\varphi)(X_{\varphi^{-1}}) = d(f \circ \varphi^{-1})(d\varphi(X_{\varphi^{-1}})).$$

□

Question: Is there an analogue to the basis of T_pM consisting of coordinate vectors at p, $\frac{\partial}{\partial x^i}\big|_p$, for local vector fields?
Answer: Yes!

Definition

Let $(\varphi = (x^1, \dots, x^n), U)$ be a chart on a smooth manifold M. The corresponding **coordinate vector fields** are defined as

$$\frac{\partial}{\partial x^i} \in \mathfrak{X}(U), \quad \frac{\partial}{\partial x^i} : p \mapsto \left.\frac{\partial}{\partial x^i}\right|_p.$$

The vector space of vector fields is infinite dimensional, hence we need to be **careful** when talking about a basis as in a basis of that vector space. We have however the following result, namely that the coordinate vector fields form a **local frame** of $TM \to M$:

Proposition B

Let $(\varphi = (x^1, \dots, x^n), U)$ be a chart on a smooth manifold M and $X \in \mathfrak{X}(U)$. With $X^i := X(x_i) \in C^\infty(U)$ we have

$$X = \sum_{i=1}^{n} X^i \frac{\partial}{\partial x^i}.$$

On the other hand for any choice of smooth functions $f^i \in C^\infty(U)$, $1 \le i \le n$,

$$\sum_{i=1}^{n} f^i \frac{\partial}{\partial x^i} \in \mathfrak{X}(U).$$

Proof: (next page)

- first equation follows from $X_p = \sum_{i=1}^{n} X_p(x^i) \left.\frac{\partial}{\partial x^i}\right|_p$ for all $p \in U$
- second equation follows from the fact that all $\frac{\partial}{\partial x^i}$ are local vector fields on U and the general fact that local sections over U (in *any* vector) bundle are a $C^\infty(U)$-module, i.e. $fs \in \Gamma(E|_U)$ for all $s \in \Gamma(E|_U)$, $f \in C^\infty(U)$ □

Examples

- $\frac{\partial}{\partial x} + r\frac{\partial}{\partial y}$, $r \in \mathbb{R}$, projected to $T^2 = \mathbb{R}^2/\mathbb{Z}^2$
- $bX \in \mathfrak{X}(M)$ for any $X \in \mathfrak{X}(U)$ and $b \in C^\infty(M)$ a bump function with support in U, $U \subset M$ open
- $\exists$ a vector field $X \in \mathfrak{X}(S^2)$ that vanishes at exactly one point, but no vector field that vanishes at no point

Remark

Proposition B means that locally, $X(f)$ for $X \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$ is of the form

$$X(f) = \sum_{i=1}^{n} X^i \frac{\partial f}{\partial x^i}.$$

One can also describe vector fields via their action on smooth functions:

Definition

Let A be an algebra over a field K. A **derivation** of a A is a K-linear map $D : A \to A$ that fulfils the Leibniz rule

$$D(ab) = D(a)b + aD(b)$$

for all $a, b \in A$. The set of all derivations of A is denoted by $\mathrm{Der}(A)$. If A is commutative, $\mathrm{Der}(A)$ is an A module.

Proposition C

Let M be a smooth manifold. Then vector fields on M are precisely the derivations of $C^\infty(M)$, meaning that $\mathfrak{X}(M)$ and $\mathrm{Der}(C^\infty(M))$ are isomorphic as $C^\infty(M)$ modules.

Proof:

- the map

$$\iota : \mathfrak{X}(M) \to \mathrm{Der}(C^\infty(M)), \quad X \mapsto (f \mapsto X(f)),$$

  is a $C^\infty(M)$ module map
- injectivity of ι follows from $X = 0$ if and only if $X(f) = 0$ for all $f \in C^\infty(M)$
- for surjectivity define for a given derivation D a vector field X^D via

$$D \mapsto X^D, \quad X^D_p(f) = D(f)(p) \quad \forall p \in M, f \in C^\infty(M)$$

- X^D is in fact a smooth vector field, and $D \mapsto X^D$ is the inverse of ι □

Assume we have two local coordinate systems with intersecting domains and write some vector field in the induced local frames. **Question:** How do we need to transform one local form to obtain the other local form? **Answer:**

Lemma

Let M be a smooth manifold and let $(\varphi = (x^1, \dots, x^n), U)$, $(\psi = (y^1, \dots, y^n), V)$ be charts on M such that $U \cap V \neq \emptyset$. For $X \in \mathfrak{X}(M)$ fixed, we have on $U \cap V$ the following forms of X in local coordinates:

$$X = \sum_{i=1}^{n} X(x^i) \frac{\partial}{\partial x^i}, \quad X = \sum_{i=1}^{n} X(y^i) \frac{\partial}{\partial y^i}.$$

We understand $d(\psi \circ \varphi^{-1}) : \varphi(U \cap V) \to \mathrm{GL}(n)$ as a **matrix-valued function** with $d(\psi \circ \varphi^{-1})_u =$ Jacobi matrix of $\psi \circ \varphi^{-1}$ at u and obtain for all $u \in U \cap V$

$$d(\psi \circ \varphi^{-1})_u \cdot \begin{pmatrix} X(x^1) \\ \vdots \\ X(x^n) \end{pmatrix} \Bigg|_{\varphi^{-1}(u)} = \begin{pmatrix} X(y^1) \\ \vdots \\ X(y^n) \end{pmatrix} \Bigg|_{\psi^{-1}(u)} .$$

END OF LECTURE 7

Next lecture:

- vector fields as a Lie algebra
- integral curves