1. The tangent bundle of a smooth manifold

2. Vector fields
Recap of lecture 6:

- recalled definition of **vector fields on** $\mathbb{R}^n$, discussed examples and action on smooth functions
- defined **vector bundles** over smooth manifolds
- defined **(local) sections** in vector bundles
- studied **transition functions** for **local trivializations** of vector bundles
- explained how to construct a vector bundle from given **vector spaces at each point** and **transition functions**
- defined term **trivializable** for vector bundles
similarly to proposition from last lecture about how to “glue” pointwise vector spaces $E_p$, $p \in M$, define:

**Definition**

Let $M$ be an $n$-dimensional smooth manifold. The **tangent bundle** $TM := \bigsqcup_{p \in M} T_p M \to M$ of $M$ with projection $\pi(v) = p$ for all $v \in T_p M$ is a vector bundle of rank $n$.

**Problem**: To show that $TM \to M$ in fact is a vector bundle, need to define local trivializations and study **matrix part** of transition functions.

**Question**: How do we get this data?

**Answer**: Use atlas and its local coordinates on $M$!
The tangent bundle $TM$ of any given manifold is, in fact, a vector bundle of rank $n$.

[Warning: There are **choices** involved!]

Proof:

- first, define candidates for **charts on the total space**
  - choose **countable atlas**
    \[ A = \{ (\varphi_i = (x^1_i, \ldots, x^n_i), U_i) \mid i \in A \} \text{ on } M \]
  - $\pi$ **smooth** by assumption $\Rightarrow \{ \pi^{-1}(U_i) \mid i \in A \}$ are **open covering** of $TM$
- define **candidates for charts** as maps between sets
  \[
  \psi_i : \pi^{-1}(U_i) \to \varphi_i(U_i) \times \mathbb{R}^n, \\
  \psi_i : v \mapsto (\varphi_i(\pi(v)), \varphi_i x^1_i, \ldots, \varphi_i x^n_i)
  \]
- observe: each $\psi_i$ is a **bijection**
define **basis of topology** on $TM$ as

$$\{\psi_i^{-1}(V) \mid i \in A, \ V \subseteq \varphi_i(U_i) \times \mathbb{R}^n \text{ open}\}$$

in this topology, all $\psi_i$ are homeomorphisms [note: this the coarsest possible topology on $TM$ so that this holds]

$M \& \mathbb{R}^n$ Hausdorff & second countable, $A$ countable

$\Rightarrow$ $TM$ with this topology also **Hausdorff & second countable**

**transition functions** of the maps $\psi_i$, $i \in A$, given by

$$\psi_i \circ \psi_j^{-1} : \varphi_j(U_i \cap U_j) \times \mathbb{R}^n \to \varphi_i(U_i \cap U_j) \times \mathbb{R}^n,$$

$$(u, w) \mapsto ((\varphi_i \circ \varphi_j^{-1})(u), d(\varphi_i \circ \varphi_j^{-1})u(w))$$

$\sim\sim$ above transition functions are **smooth**, hence $\psi_i$, $i \in A$, form an $2n$-**dimensional smooth atlas** on total space $TM$
corresponding candidates for **local trivializations**

\[(\varphi_i^{-1} \times \text{id}_{\mathbb{R}^n}) \circ \psi_i\]

of \(TM\) are of the form

\[
\tau_{ij} := (\varphi_i^{-1} \times \text{id}_{\mathbb{R}^n}) \circ \psi_i \circ \left( (\varphi_j^{-1} \times \text{id}_{\mathbb{R}^n}) \circ \psi_j \right)^{-1}
\]

\[= (\text{id}_M, d(\varphi_i \circ \varphi_j^{-1}))\]

all \(\tau_{ij} : (U_i \cap U_j) \times \mathbb{R}^n \to (U_i \cap U_j) \times \mathbb{R}^n\) are smooth and of the form \((p, v) \mapsto A(p)v, A : U_i \cap U_j \to \text{GL}(n)\)

hence: \(TM\) with the so-defined local trivializations is a vector bundle as claimed

**Remark:** \(\tau_{ij}\) and local trivializations **compatible** in sense of proposition about constructing vector bundles from local bijections and given smooth transition functions, see Lecture 6.

\(\leadsto\) still abstract description of \(TM\), locally we have a simpler picture: (next page)
Remark

Let \( \varphi = (x^1, \ldots, x^n) \) be a local coordinate system on \( M \) covering \( p \in M \). Let \( v \in T_p M \),

\[
v = \sum_{i=1}^{n} v^i \left. \frac{\partial}{\partial x^i} \right|_p,
\]

be arbitrary. For

\[
\psi = (\varphi \circ \pi, d\varphi)
\]
as in the previous Proposition A we get

\[
\psi(v) = (x^1(p), \ldots, x^n(p), v^1, \ldots, v^n),
\]

meaning that the vector part of \( \psi(v) \) consists of the prefactors of \( v \) in the basis \( \left\{ \left. \frac{\partial}{\partial x^i} \right|_p, \ 1 \leq i \leq n \right\} \).
Example

Let \( M \subset \mathbb{R}^n \) be a smooth submanifold of codimension \( 0 < k < n \). Locally, write \( M \) as graph of a smooth map

\[
g : U \subset \mathbb{R}^{n-k} \to \mathbb{R}^k.
\]

Pointwise, \( T_pM \) is the set of tangent vectors \( v \in T_p\mathbb{R}^n \) that are in the image of \( dg_p \). The induced chart on \( M \) is given by

\[
\varphi : \text{graph}(g) \to \mathbb{R}^{n-k}, \quad (p, g(p)) \mapsto p,
\]

and the inverse of the induced local trivialization \( \psi \) of \( TM \) is given by

\[
\psi^{-1} : U \times \mathbb{R}^{n-k} \to TM|_U, \quad (p, v) \mapsto ((p, g(p)), (v, dg_p(v))).
\]

Question: Can we imagine the tangent bundle, locally, of any smooth manifold as in the above example?

Answer: Yes! (next page)
The tangent bundle of a smooth manifold

**Theorem (Strong Whitney Embedding Theorem)**

*Every* smooth manifold of dimension $n$ can be realized as an *embedded submanifold* of $\mathbb{R}^{2n}$.

**Proof:** See Theorem 6.19 in “*Introduction to Smooth Manifolds*” (John M. Lee), Springer GTM 218.

Recall that $TM$ is called *trivializable* if it is isomorphic as a vector bundle to $M \times \mathbb{R}^n$, $n = \dim(M)$.

**Lemma**

$TS^1$, $TS^3$, and $TS^7$ are trivializable.

**Proof:** $S^1$:

- view $S^1 = \{z \bar{z} = 1\} \subset \mathbb{C} \cong \mathbb{R}^2$
- for $z \in S^1$ fixed, $iz$ spans $T_zS^1$
- define vector bundle isomorphism

\[ F : S^1 \times \mathbb{R} \to TS^1, \quad (z, t) \mapsto (z, tiz) \]
hence: total space of $TS^1 \to S^1$ is diffeomorphic to the cylinder

- $S^3$ & $S^7$: Exercise! [hint: view $S^3$ as the unit quaternions and $S^7$ as the unit octonions]

Now we have all tools at hand to define vector fields on smooth manifolds.

**Definition**

Sections in the tangent bundle of a smooth manifold, $\Gamma(TM)$, are called **vector fields**. For $X \in \Gamma(TM)$ we will denote the value of $X$ at $p \in M$ by $X_p$. For $U \subset M$ open, we will call elements of $\Gamma(TM|_U)$ **local vector fields**, or simply vector fields if the setting does not explicitly use the locality property. We will use the notations

$$\mathfrak{X}(M) := \Gamma(TM)$$

and

$$\mathfrak{X}(U) := \Gamma(TM|_U)$$

for $U \subset M$ open.
Vector fields

Remark
For a smooth manifold $M$ and $U \subset M$ open, the two vector spaces $T_pU$ and $T_pM$ are canonically isomorphic via restriction of charts for all $p \in U$.

If so-defined vector fields look a bit alien at first glance, it helps to look at them through the lens of local coordinates:

Remark
In local coordinates $(\varphi = (x^1, \ldots, x^n), U)$ on a smooth manifold $M$ with induced coordinates $\psi = (\varphi \circ \pi, d\varphi)$ on $TM|_U \subset TM$, (local) vector fields $X$ have coordinate representations of the form

$$\hat{X} : \varphi(U) \to U \times \mathbb{R}^n, \quad \hat{X} = (\text{id}_U, d\varphi(X\varphi^{-1})),$$

$\hat{X}$ smooth. This means that locally up to the explicit notation of the base point, vector fields on smooth manifolds are vector fields on an open subset of $\mathbb{R}^n$.
Vector fields, similar to tangent vectors, act on $C^\infty(M)$ by

$$X(f)(p) := X_p(f) = df(X_p).$$

Thus we may write $X(f) = df(X) \in C^\infty(M)$. On the other hand, a map of the form $X : M \to TM$, $p \mapsto X_p \in T_pM$, is a vector field if $X(f) : p \mapsto df_p(X_p)$ is smooth for all $f \in C^\infty(M)$.

The above is **compatible** with writing $X$ and $f$ in coordinate representations:

**Lemma**

The coordinate representation of $X(f)$ is the same as the action of the vector part of the coordinate representation of $X$ on the coordinate representation of $f$.

**Proof:** (next page)
(continuation of proof)
Write \( \hat{f} = f \circ \varphi^{-1} \), vector part of \( \hat{X} = d\varphi(X_{\varphi^{-1}}) \). We obtain

\[
X(f) \circ \varphi^{-1} = df(X) \circ \varphi^{-1} \\
= (d(f \circ \varphi^{-1}) \circ d\varphi)(X_{\varphi^{-1}}) = d(f \circ \varphi^{-1})(d\varphi(X_{\varphi^{-1}})).
\]

**Question:** Is there an analogue to the basis of \( T_pM \) consisting of coordinate vectors at \( p \), \( \frac{\partial}{\partial x^i} \bigg|_p \), for local vector fields?

**Answer:** Yes!

**Definition**
Let \( (\varphi = (x^1, \ldots, x^n), U) \) be a chart on a smooth manifold \( M \). The corresponding **coordinate vector fields** are defined as

\[
\frac{\partial}{\partial x^i} \in \mathfrak{X}(U), \quad \frac{\partial}{\partial x^i} : p \mapsto \left. \frac{\partial}{\partial x^i} \right|_p.
\]
The vector space of vector fields is infinite dimensional, hence we need to be careful when talking about a basis as in a basis of that vector space. We have however the following result, namely that the coordinate vector fields form a local frame of $TM \to M$:

**Proposition B**

Let $(\varphi = (x^1, \ldots, x^n), U)$ be a chart on a smooth manifold $M$ and $X \in \mathfrak{X}(U)$. With $X^i := X(x_i) \in C^\infty(U)$ we have

$$X = \sum_{i=1}^{n} X^i \frac{\partial}{\partial x^i}.$$

On the other hand for any choice of smooth functions $f^i \in C^\infty(U), 1 \leq i \leq n$,

$$\sum_{i=1}^{n} f^i \frac{\partial}{\partial x^i} \in \mathfrak{X}(U).$$

**Proof:** (next page)
first equation follows from $X_p = \sum_{i=1}^{n} X_p(x^i) \frac{\partial}{\partial x^i}|_p$ for all $p \in U$

second equation follows from the fact that all $\frac{\partial}{\partial x^i}$ are local vector fields on $U$ and the general fact that local sections over $U$ (in any vector) bundle are a $C^\infty(U)$-module, i.e. $fs \in \Gamma(E|_U)$ for all $s \in \Gamma(E|_U)$, $f \in C^\infty(U)$

Examples

- $\frac{\partial}{\partial x} + r \frac{\partial}{\partial y}$, $r \in \mathbb{R}$, projected to $T^2 = \mathbb{R}^2/\mathbb{Z}^2$
- $bX \in \mathfrak{X}(M)$ for any $X \in \mathfrak{X}(U)$ and $b \in C^\infty(M)$ a bump function with support in $U$, $U \subset M$ open
- $\exists$ a vector field $X \in \mathfrak{X}(S^2)$ that vanishes at exactly one point, but no vector field that vanishes at no point
Vector fields

**Remark**

Proposition B means that locally, \( X(f) \) for \( X \in \mathfrak{X}(M) \) and \( f \in C^\infty(M) \) is of the form

\[
X(f) = \sum_{i=1}^{n} X^i \frac{\partial f}{\partial x^i}.
\]

One can also describe vector fields via their action on smooth functions:

**Definition**

Let \( A \) be an algebra over a field \( K \). A **derivation** of a \( A \) is a \( K \)-linear map \( D : A \to A \) that fulfils the Leibniz rule

\[
D(ab) = D(a)b + aD(b)
\]

for all \( a, b \in A \). The set of all derivations of \( A \) is denoted by \( \text{Der}(A) \). If \( A \) is commutative, \( \text{Der}(A) \) is an \( A \) module.
**Proposition C**

Let $M$ be a smooth manifold. Then vector fields on $M$ are precisely the derivations of $C^\infty(M)$, meaning that $\mathfrak{X}(M)$ and $\text{Der}(C^\infty(M))$ are isomorphic as $C^\infty(M)$ modules.

**Proof:**

- the map 
  \[ \iota : \mathfrak{X}(M) \to \text{Der}(C^\infty(M)), \quad X \mapsto (f \mapsto X(f)), \]
  is a $C^\infty(M)$ module map

- injectivity of $\iota$ follows from $X = 0$ if and only if $X(f) = 0$ for all $f \in C^\infty(M)$

- for surjectivity define for a given derivation $D$ a vector field $X^D$ via 
  \[ D \mapsto X^D, \quad X^D_p(f) = D(f)(p) \quad \forall p \in M, f \in C^\infty(M) \]

- $X^D$ is in fact a smooth vector field, and $D \mapsto X^D$ is the inverse of $\iota$
Assume we have two local coordinate systems with intersecting domains and write some vector field in the induced local frames.

**Question:** How do we need to transform one local form to obtain the other local form?  

**Answer:**

**Lemma**

Let \( M \) be a smooth manifold and let \( (\varphi = (x^1, \ldots, x^n), U) \), \( (\psi = (y^1, \ldots, y^n), V) \) be charts on \( M \) such that \( U \cap V \neq \emptyset \). For \( X \in \mathfrak{X}(M) \) fixed, we have on \( U \cap V \) the following forms of \( X \) in local coordinates:

\[
X = \sum_{i=1}^{n} X(x^i) \frac{\partial}{\partial x^i}, \quad X = \sum_{i=1}^{n} X(y^i) \frac{\partial}{\partial y^i}.
\]

We understand \( d(\psi \circ \varphi^{-1}) : \varphi(U \cap V) \to \text{GL}(n) \) as a **matrix-valued function** with \( d(\psi \circ \varphi^{-1})_u = \text{Jacobi matrix of } \psi \circ \varphi^{-1} \) at \( u \) and obtain for all \( u \in U \cap V \)

\[
d(\psi \circ \varphi^{-1})_u \cdot 
\begin{pmatrix}
X(x^1) \\
\vdots \\
X(x^n)
\end{pmatrix}
_{\varphi^{-1}(u)} = 
\begin{pmatrix}
X(y^1) \\
\vdots \\
X(y^n)
\end{pmatrix}
_{\psi^{-1}(u)}.
\]
END OF LECTURE 7

Next lecture:
- vector fields as a Lie algebra
- integral curves