Differential geometry
Lecture 6: Vector bundles

David Lindemann

University of Hamburg
Department of Mathematics
Analysis and Differential Geometry & RTG 1670

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Recap of lecture 5:

- **inverse function theorem** for smooth manifolds
- characterisation of local diffeomorphisms
- defined **(embedded) smooth submanifolds**
- showed that locally we can always find **adapted coordinates**
- discussed **rank theorem**
- proved that level sets consisting only of **regular points** are smooth submanifolds
- erratum: not really an error, but forgot to define **smooth hypersurfaces**

At this point, we know what tangent vectors on smooth manifolds are.
Question: How should we define **vector fields** on smooth manifolds? $\rightsquigarrow$ "smoothly varying tangent vectors"
Recall $\mathbb{R}^n$ case:

Definition

- a **vector field** X on $\mathbb{R}^n$ is a smooth vector valued function $X : \mathbb{R}^n \to \mathbb{R}^n$, $p \mapsto X(p) = X_p$
- points $(p, X_p) \in \mathbb{R}^n \times \mathbb{R}^n$ "=" **tangent vector** X_p with **basepoint** p
- vector fields act on smooth functions $f \in C^\infty(\mathbb{R}^n)$ via

$$X(f) := df(X), \quad p \mapsto df_p(X_p)$$

- pointwise: action comes from action of $X_p \in T_p\mathbb{R}^n \cong \mathbb{R}^n$ on f

Examples

- the **position vector field** $X : p \mapsto p \; \forall p \in \mathbb{R}^n$
- $X : p \mapsto v$ for $v \in \mathbb{R}^n$ fixed
- on $\mathbb{R}^2$: $X : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -y \\ x \end{pmatrix}$

Additional question: Should the second example be called a "constant vector field" (because each entry is constant)?
Answer: Later, needs definition of **connections** (**covariant derivatives**).

To make sense of vector fields on general smooth manifold, need the following concept:

Definition

A **vector bundle** $E \to M$ **of rank** $k \in \mathbb{N}$ over a smooth manifold M is a smooth manifold E together with a smooth **projection map** $\pi : E \to M$, such that

- the **fibre** $E_p := \pi^{-1}(p)$ is an k-dimensional real vector space for all $p \in M$,
- for all $p \in M$ $\exists$ open neighbourhood $U \subset M$ of p and a diffeomorphism $\psi : \pi^{-1}(U) \to U \times \mathbb{R}^k$, such that $\psi|_{E_q} : E_q \to q \times \mathbb{R}^k \cong \mathbb{R}^k$ is a linear isomorphism for all $q \in U$ and the diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\quad\psi\quad} & U \times \mathbb{R}^k \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\mathrm{pr}_U} & \\ & U & \end{array}$$

  commutes. The map pr_U denotes the canonical projection onto the first factor. *(continued on next page)*

E is called the **total space**, M is called the **basis**, and the map ψ is called a **local trivialization** of the vector bundle $E \to M$.

Example

$M \times \mathbb{R}^k \to M$, $\pi(p, v) := p \ \forall p \in M, v \in \mathbb{R}^k$ **[**for $k \in \mathbb{N}_0$ fixed**]**

A generalisation of vector valued functions on $\mathbb{R}^n$ are sections in vector bundles:

Definition

A **local section in a vector bundle** $E \to M$ is a smooth map

$$s : U \to E$$

$U \subset M$ open, such that $\pi \circ s = \mathrm{id}_U$, that is $s(p) \in E_p \ \forall p \in U$. If $U = M$, s is called a **(global) section**. The set of local sections in $E \to M$ on $U \subset M$ is denoted by $\Gamma(E|_U)$ and the set of global sections by $\Gamma(E)$, where $E|_U$ denotes the vector bundle $\pi^{-1}(U) \to U$. The **support** of a section (or, analogously, local section) in a vector bundle $s \in \Gamma(E)$ is defined to be the set

$$\mathrm{supp}(s) := \overline{\{p \in M \mid s(p) \neq 0\}}.$$

Note: $\Gamma(E)$ is a $C^\infty(M)$-module, $\Gamma(E|_U)$ is a $C^\infty(U)$-module for $U \subset M$ open.

Similar to transition function for charts in an atlas there are transition functions for the local trivializations of vector bundles:

Definition

Let $\psi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ and $\phi : \pi^{-1}(V) \to V \times \mathbb{R}^k$ be two local trivializations of a vector bundle $E \to M$ with $U \cap V \neq \emptyset$. Then the smooth map

$$(\psi \circ \phi^{-1}) : (U \cap V) \times \mathbb{R}^k \to (U \cap V) \times \mathbb{R}^k$$

is called **transition function**. For $p \in M$ fixed, $(\psi \circ \phi^{-1})(p, \cdot)$ is called **transition function at** p.

Lemma

Transition functions of vector bundles are of the form

$$\psi \circ \phi^{-1} : (p, v) \mapsto (p, A(p)v), \quad A(p) \in \mathrm{GL}(k),$$

for all $p \in U \cap V$, $v \in \mathbb{R}^n$. The map

$$A : U \cap V \to \mathrm{GL}(k), \quad p \mapsto A(p),$$

is **smooth**.

Proof:

- following diagram commutes:

$$\begin{array}{ccccc} U \cap V \times \mathbb{R}^k & \xrightarrow{\phi^{-1}} & \pi^{-1}(U \cap V) & \xrightarrow{\psi} & U \cap V \times \mathbb{R}^k \\ & \searrow^{\mathrm{pr}_{U\cap V}} & \downarrow \pi & \swarrow_{\mathrm{pr}_{U\cap V}} & \\ & & U \cap V & & \end{array}$$

- hence: $\psi \circ \phi^{-1}$ sends (p, v) to $(p, A(p, v))$ for **some** smooth function $A : U \cap V \times \mathbb{R}^k \to \mathbb{R}^k$
- smoothness follows from ϕ and ψ being diffeos
- still need to show that for p fixed, $A(p, \cdot) : \mathbb{R}^k \to \mathbb{R}^k$ is an **invertible linear map**
- above follows from the fact that fibre-wise ϕ and ψ are **linear** isomorphisms □

Now suppose we are given not a vector bundle, but at each point p in a manifold M a **vector space of fixed dimension** E_p, **local trivialisations** as maps between sets (pointwise linear), and their **"transition functions"** behave like transition functions of a vector bundle.

Question: Can we use the above data to find a **vector bundle structure** on the disjoint union

$$E := \bigsqcup_{p \in M} E_p \quad ?$$

Answer: Yes!

Proposition

Let M be a smooth manifold and assume that for every $p \in M$, E_p is a real vector space of **fixed dimension** k. Define a set

$$E := \bigsqcup_{p \in M} E_p$$

and a map $\pi : E \to M$, $\pi(v) = p\ \forall v \in E_p$ and all $p \in M$. Assume that $\{U_i,\ i \in I\}$ is an **open cover** of M and

$$\phi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^k$$

is a **bijection** $\forall i \in I$ such that $\phi_i : E_p \to \{p\} \times \mathbb{R}^k \cong \mathbb{R}^k$ is a **linear isomorphism** $\forall p \in M$. Further assume that $\forall i, j \in I$ with $U_i \cap U_j \neq \emptyset\ \exists$ a **smooth map** $\tau_{ij} : U_i \cap U_j \to \mathrm{GL}(k)$, such that $\phi_i \circ \phi_j^{-1} : (U_i \cap U_j) \times \mathbb{R}^k \to (U_i \cap U_j) \times \mathbb{R}^k$ is of the form

$$\phi_i \circ \phi_j^{-1}(p, v) = (p, \tau_{ij}(p)v).$$

Then there exists a **unique topology** and **maximal atlas** on E, such that $\pi : E \to M$ is a **vector bundle of rank** k and the ϕ_i, $i \in I$, are local trivializations.

Proof:

- wlog assume that we can find an atlas $\{(\varphi_i, U_i) \mid i \in I\}$ on M
- can always be achieved by **shrinking** the U_i if necessary and, on possible new overlaps $U_i \cap U_j$, set $\tau_{ij} \equiv \mathrm{id}_{\mathbb{R}^k}$
- $\rightsquigarrow$ can explicitly construct an atlas on the total space E:
- define for $i \in I$

$$\psi_i : \pi^{-1}(U_i) \to \varphi_i(U_i) \times \mathbb{R}^k, \quad v \mapsto (\varphi_i \times \mathrm{id}_{\mathbb{R}^k})(\phi_i(v)).$$

- for $\{(\psi_i, \pi^{-1}(U_i)) \mid i \in I\}$ to be a **smooth atlas** on E, need to show that the **transition functions** (as in transition functions of a smooth atlas) are smooth
- check:

$$\psi_i(\pi^{-1}(U_i) \cap \pi^{-1}(U_j)) = \varphi_i(U_i \cap U_j) \times \mathbb{R}^k$$

for all $i, j \in I$

- calculate:

$$\psi_i \circ \psi_j^{-1} = (\varphi_i \times \mathrm{id}_{\mathbb{R}^k}) \circ (\phi_i \circ \phi_j^{-1}) \circ (\varphi_j^{-1} \times \mathrm{id}_{\mathbb{R}^k})$$
$$: \varphi_j(U_i \cap U_j) \times \mathbb{R}^k \to \varphi_i(U_i \cap U_j) \times \mathbb{R}^k.$$

- by assumption $\tau_{ij}(p)$ is invertible and depends smoothly on $p \in U_i \cap U_j$
- $\rightsquigarrow \phi_i \circ \phi_j^{-1}$ is a diffeomorphism for all $i, j \in I$ such that $U_i \cap U_j \neq \emptyset$
- since $\{(\varphi_i, U_i),\ i \in I\}$ is smooth atlas on M, each $\varphi_i \times \mathrm{id}_{\mathbb{R}^k}$ is a diffeo
- hence: $\psi_i \circ \psi_j^{-1} : \varphi_j(U_i \cap U_j) \times \mathbb{R}^k \to \varphi_i(U_i \cap U_j) \times \mathbb{R}^k$ is also a diffeo for all $i, j \in I$, such that $U_i \cap U_j \neq \emptyset$
- defining the open sets on E as the preimages of open sets under ψ_i, $i \in I$, get that it is **second countable** and **Hausdorff** by assumption that M and $\mathbb{R}^k$ are smooth manifolds

- equipped with the so-defined topology, $\mathcal{B} := \{(\psi_i, \pi^{-1}(U_i)) \mid i \in I\}$ is a **smooth atlas on the total space** E
- $\rightsquigarrow$ all maps ϕ_i, $i \in I$, are **automatically smooth**
- furthermore, since $\phi_i : E_p \to \{p\} \times \mathbb{R}^k$ is a linear isomorphism by assumption, ϕ_i, $i \in I$, form a covering of local trivializations turning $E \to M$ into a **vector bundle of rank** k
- **uniqueness** of the smooth manifold structure on E now follows from the assumption that all ϕ_i are diffeomorphisms onto their image and, thus, every smooth atlas on E with that property must, by construction, be a refinement of $\mathcal{B}$ and thus be contained in the same maximal smooth atlas as $\mathcal{B}$ □

Question: What should a homomorphism of vector bundles fulfil? **Answer:**

Definition

Let $\pi_E : E \to M$ and $\pi_F : F \to M$ be vector bundles over smooth manifolds M. Then a **smooth vector bundle homomorphism**[a] is a smooth map between the total spaces

$$f : E \to F,$$

such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{\quad f \quad} & F \\ & {\scriptstyle \pi_E} \searrow \qquad \swarrow {\scriptstyle \pi_F} & \\ & M & \end{array}$$

commutes and f is fibrewise linear. The last condition means that for each $p \in M$, $f|_{E_p} : E_p \to F_p$ is a linear map.

[a] a.k.a. "smooth vector bundle map"

This enables us to define when two vector bundles are **isomorphic**:

Definition

Two vector bundles $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M$ are **isomorphic** if there exists a diffeomorphism $F : E_1 \to E_2$ that is a smooth vector bundle map, so that

$$F|_{E_{1p}} : E_{1p} \to E_{2p}$$

is a linear isomorphism for all $p \in M$.

The “best case scenario” (as in “easiest to deal with”) is the following:

Definition

A vector bundle of rank k, $E \to M$, is called **trivializable** if it is isomorphic to $M \times \mathbb{R}^k \to M$ equipped with the canonical projection onto M.

If a vector bundle is trivializable we have the following result:

Lemma

Assume that $E \to M$ is trivializable. Then there exists a **nowhere vanishing** global section $s \in \Gamma(E)$.

Proof: Exercise.

END OF LECTURE 6

Next lecture:

- tangent bundle
- vector fields on smooth manifolds