Differential geometry
Lecture 5: Submanifolds

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5. May 2020

Recap of lecture 4:

- explained how to write down tangent vectors explicitly with the help of local coordinates
- showed that coordinate tangent vectors $\frac{\partial}{\partial x^i}\Big|_p$ for a basis of T_pM
- defined differential, Jacobi matrix, and rank of smooth maps
- defined immersions, submersions, embeddings, local diffeomorphisms
- erratum: slight confusion in definition of local diffeomorphisms

Question: How can we check if a smooth map is a local diffeomorphism?
Answer: Need following theorem:

Theorem

Let $F : M \to N$ be a smooth map, $\dim(M) = \dim(N) = n$, and $p \in M$. $dF_p : T_pM \to T_{F(p)}N$ is a linear isomorphism **if and only if** $\exists$ open nbh. $U \subset M$ of p, such that

$$F|_U : U \to F(U) \subset N$$

is a diffeomorphism.

Proof:

- choose local charts (φ, U) and (ψ, V) of M, N, covering p, $F(p)$, respectively
- observation 1: dF_p is a linear isomorphism if and only if the Jacobi matrix of F in local coordinates φ, ψ, is invertible

(continued on next page)

- observation 2: $\exists$ open nbh. $U \subset M$ of p, such that $F|_U : U \to F(U)$ is invertible if an only if $\exists\, U', V' \subset \mathbb{R}^n$ open with $\varphi(p) \in U'$, $\psi(F(p)) \in V'$, such that

$$\psi \circ F \circ \varphi^{-1} : U' \to V'$$

is a diffeomorphism
- "$\Rightarrow$" follows from the inverse function theorem from real analysis
- "$\Leftarrow$" follows from fact that diffeomorphisms in the real analysis sense have pointwise invertible Jacobi matrix □

This characterises local diffeomorphisms as follows:

Corollary

$F : M \to N$ is a local diffeomorphism **if and only if** dF_p is a linear isomorphism for all $p \in M$.

Examples

- The projection map

$$\pi : S^n \to \mathbb{R}P^n, \quad (x^1, \dots, x^n) \mapsto [x^1 : \dots : x^n]$$

is a local diffeomorphism $\forall\ n \in N$ but never a diffeomorphism.

- There are examples for local diffeomorphism for M non-compact and N compact, e.g.

$$f : \mathbb{R} \to S^1 \cong \mathbb{R}/\mathbb{Z}, \quad t \mapsto [t],$$

where $[t] = [t']$ if $\exists\ z \in \mathbb{Z}$, such that $t = t' + z$.

Recall: Smooth submanifolds of $\mathbb{R}^n$ defined as subsets $S \subset \mathbb{R}^n$ that be locally written as

$$S \cap U = \{f(p) = 0 \mid p \in U\},$$

$U \subset \mathbb{R}^n$ open, $f : U \to \mathbb{R}^m$ with Jacobi matrix of maximal rank in U. $\rightsquigarrow$ use **embeddings** to generalize this concept to smooth manifold:

Definition

Let N be an n-dim., M be an m-dim. smooth manifold and $F : M \to N$ a smooth map.

- $F(M) \subset N$ is called an **embedded smooth submanifold** if F is an embedding.
- If F is the inclusion map $\iota : M \hookrightarrow N$, we will say that $M \subset N$ is a **smooth submanifold** if the inclusion is an embedding.
- If $M \subset N$ is a smooth submanifold, the number $\dim(N) - \dim(M)$ is called the **codimension** of M in N.

Question: is there a "good" choice of charts on N w.r.t. some submanifold $M \subset N$?

Proposition A

Let $M \subset N$, $\dim(M) = m < n = \dim(N)$, be a smooth submanifold and let $p \in M$ be arbitrary. Then there exists a chart $(\varphi = (x^1, \ldots, x^n), U)$ on N, such that $U \cap M$ is an open neighbourhood of p in M and

$$x^{m+1}(q) = \ldots = x^n(q) = 0$$

for all $q \in U \cap M$. The first m entries in φ are a local coordinate system on M near p.

Proof:

- fix $p \in M \subset N$, choose local coordinates $(x^1, \ldots, x^n)$ on N and $(y^1, \ldots, y^m)$ on M covering p
- M being a submanifold means inclusion $\iota : M \to N$ is an embedding, thus $d\iota_p$ is in particular injective

- hence, the Jacobi matrix of ι at p in local coordinates φ, ψ,
$$\left(\frac{\partial x^i}{\partial y^j}(p)\right)_{ij} \in \mathrm{Mat}(n \times m, \mathbb{R})$$
has maximal rank m
- w.l.o.g. (after possibly re-ordering the x^i) assume that first m rows are linearly independent
- implicit function theorem $\Rightarrow (x^1, \dots, x^m)$ are local coordinates on some open set $V \subset M$
- after possibly shrinking V obtain that $q \in \iota(M)$ iff
$$x^k(q) = f^k(x^1(q), \dots, x^m(q)),$$
$f^k : (x^1, \dots, x^m)(V) \to \mathbb{R}$ uniquely determined $\forall$ $m + 1 \leq k \leq n$
- choose $U \subset N$ open, such that $(x^1, \dots, x^n)$ are defined on U and $U \cap M = V$, define for $m + 1 \leq k \leq n$
$$F^k := x^k - f^k(x^1, \dots, x^m)$$

- define new coordinate system (φ, U) on N fulfilling statement of this proposition as follows:

$$\varphi = (x^1, \dots, x^m, F^{m+1}, \dots, F^n)$$

- Jacobi matrix of φ at p with respect to the coordinates $(x^1, \dots, x^n)$ is of the form

$$\begin{pmatrix} \mathrm{id}_{\mathbb{R}^m} & 0 \\ A & \mathrm{id}_{\mathbb{R}^{n-m}} \end{pmatrix}$$

  for some $A \in \mathrm{Mat}((n-m) \times m, \mathbb{R})$, hence invertible

- hence, φ is a local diffeomorphism and thereby defines a local coordinate system on N which, by construction, fulfils

$$\varphi(U \cap M) = \varphi(V) = (x^1, \dots, x^m, 0, \dots, 0).$$

  as required □

Definition

Local coordinates as in Proposition A for a submanifold $M \subset N$ near a given point $p \in M$ are called **adapted coordinates**.

From the proof of Proposition A we obtain:

Corollary

Any smooth manifold M that can be realized as a submanifold of some ambient manifold N is diffeomorphic to M, viewed as a topological subspace of N, equipped with any atlas consisting only of adapted coordinates.

Next: Study methods to obtain examples of smooth submanifolds.

Definition

Let M and N be smooth manifolds and let $F : M \to N$ be a smooth map.

- $p \in M$ is called **regular point** of F if $dF_p : T_pM \to T_{F(p)}N$ is surjective
- $q \in N$, such that $F^{-1}(q) \subset M$ consists only of regular points, is called **regular value** of F
- points in M that are not regular points of F are called **critical points** of F
- points in N such that the pre-image under F in M contains at least one critical point of F are called **critical values** of F

Remark: $F : M \to N$ can only have regular values if $\dim(M) \geq \dim(N)$

A convenient way to construct smooth submanifold is the following:

Proposition B

Let $F : M \to N$ be smooth, $\dim(M) = m \geq n = \dim(N)$, and $q \in N$ a regular value of F. Then the level set

$$F^{-1}(q) \subset M$$

is an $(m-n)$-dimensional smooth submanifold of M. The structure of a smooth manifold on $F^{-1}(q)$ is uniquely determined by requiring that the inclusion is smooth.

$\rightsquigarrow$ in order to prove Proposition B, need additional tools

Definition

Let $F : M \to N$ be a smooth, (φ, U) and (ψ, V) be local charts of M and N, respectively, such that $F(U) \subset V$, and $\dim(M) = m$ and $\dim(N) = n$. The **coordinate representation of** F in the local coordinate systems φ and ψ is defined to be the smooth map

$$\widehat{F} : \varphi(U) \to \psi(V),$$

$$\widehat{F}(u^1, \dots, u^m) := (\psi \circ F \circ \varphi^{-1})(u^1, \dots, u^m).$$

Using the above definition, we can formulate the so-called **rank theorem**: (next page)

Theorem A

Let M be an m-dim. and N be an n-dim. smooth manifold. Let $F : M \to N$ be a smooth map of constant rank r. Then for each $p \in M$ $\exists$ local charts (φ, U) of M with $p \in U$ and (ψ, V) of N with $F(p) \in V$, such that $F(U) \subset V$ and that the coordinate representation of F is of the form

$$\widehat{F}(u^1, \dots, u^r, u^{r+1}, \dots, u^m) = (u^1, \dots, u^r, 0, \dots, 0).$$

Proof: See Thm. 4.12 in *Introduction to Smooth Manifolds* (John M. Lee), or in the real analysis setting with a slightly different formulation Thm. 9.32 in *Principles of Mathematical Analysis* (W. Rudin).

Theorem B

Let M and N be smooth manifolds and $F : M \to N$ smooth and of constant rank r. Each level set $F^{-1}(q) \subset M$, $q \in N$, is a smooth submanifold of codimension r in M.

Proof:

- fix $q \in N$ and $p \in F^{-1}(q)$
- Theorem A $\rightsquigarrow$ choose charts $(\varphi = (x^1, \ldots, x^m), U)$ of M, $p \in U$, and (ψ, V) of N, $q \in V$, fulfilling

  $$\varphi(p) = 0, \quad \psi(q) = 0,$$

  such that the coordinate representation $\widehat{F}$ of F is of the form

  $$\widehat{F} : \varphi(U) \to \psi(V),$$
  $$\widehat{F}(u^1, \ldots, u^r, u^{r+1}, \ldots, u^m) = (u^1, \ldots, u^r, 0, \ldots, 0)$$

- obtain $(\psi \circ F)(x^1, \dots, x^m) = (x^1, \dots, x^r, 0, \dots, 0)$
- this shows that

$$F^{-1}(q) \cap U = \{p \in U \mid x^1(p) = \dots = x^r(p) = 0\}$$

- hence, $(x^{r+1}, \dots, x^m)$ are (up to reordering) adapted coordinates on $F^{-1}(q) \cap M$
- cover $F^{-1}(q)$ with so constructed coordinates, obtain that $F^{-1}(q) \subset M$ is, in fact, a submanifold of M
- every max. atlas on $F^{-1}(q)$, such that the inclusion is smooth, contains all so constructed adapted coordinates
- hence structure of a smooth manifold on $F^{-1}(q)$, such that the inclusion is smooth, is unique $\square$

Need one more result:

Proposition C

Let $F : M \to N$ be smooth. The set of regular points of F is open in M.

Proof: Exercise!

Using Theorem B and Proposition C we can finally prove Proposition C:

Proof (of Proposition C):

- let $F : M \to N$ be smooth, $q \in N$ a regular value of F
- Proposition C $\rightsquigarrow$

$$\mathrm{reg}(F) := \{p \in M \mid p \text{ regular point of } F\}$$

open in M

- by assumption we have $F^{-1}(q) \subset \mathrm{reg}(F)$, hence

$$F|_{\mathrm{reg}(F)} : \mathrm{reg}(F) \to N$$

  is a submersion and, hence, of constant rank equal to $\dim(N)$
- Theorem B $\rightsquigarrow$ $F^{-1}(q) \subset \mathrm{reg}(F)$ is a smooth submanifold
- the composition of the inclusions $F^{-1}(q) \subset \mathrm{reg}(F)$ and $\mathrm{reg}(F) \subset M$ is still the inclusion, in particular still a smooth embedding
- hence $F^{-1}(q) \subset M$ is a smooth submanifold
- $\mathrm{reg}(F) \subset M$ open and Theorem B imply $\dim(F^{-1}(q)) = m - n$
- structure of a smooth manifold on $F^{-1}(q)$ uniquely determined by requiring that the inclusion in M is smooth by Theorem B □

END OF LECTURE 5

Next lecture:

- vector bundles