Differential geometry
Lecture 4: Tangent spaces (part 2)

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Recap of lecture 3:

- defined **vector space of smooth functions** $C^\infty(M)$
- defined **bump functions** (w.r.t. given data)
- recalled **tangent vectors & tangent space of** $\mathbb{R}^n$
- defined **tangent vectors for general smooth manifolds**
- showed that tangent vectors are **local objects**
- erratum: **nope!**

How can we write down tangent vectors explicitly? Use **local coordinates**!

Definition

Let $\varphi = (x^1, \dots, x^n)$ be local coordinates on a smooth manifold M defined on $U \subset M$. Then $\frac{\partial}{\partial x^i}\big|_p \in T_pM$, $p \in U$, is defined as

$$\frac{\partial}{\partial x^i}\bigg|_p (f) := \frac{\partial f}{\partial x^i}(p) := \frac{\partial (f \circ \varphi^{-1})}{\partial u^i}(\varphi(p)) \quad \forall f \in C^\infty(M).$$

- we can think of $\frac{\partial}{\partial x^i}\big|_p$ as partial derivative at p with respect to the chosen local coordinates $(x^1, \dots, x^n)$
- alternative notations: $\partial_i\big|_p$, $\partial_{x^i}\big|_p$

$\leadsto$ need to check the following:

Lemma

$\frac{\partial}{\partial x^i}\big|_p$ is a well-defined tangent vector.

Proof: (next page)

$\rightsquigarrow$ need to check **linearity** and **Leibniz rule**:

- linearity follows from linearity of partial derivatives in $\mathbb{R}^n$
- Leibniz rule: for $f, g \in C^\infty(M)$ we calculate

$$\begin{aligned}
&\frac{\partial(f \cdot g)}{\partial x^i}(p) \\
&= \frac{\partial((f \cdot g) \circ \varphi^{-1})}{\partial u^i}(\varphi(p)) \\
&= \frac{\partial((f \circ \varphi^{-1}) \cdot (g \circ \varphi^{-1}))}{\partial u^i}(\varphi(p)) \\
&= g(p)\frac{\partial(f \circ \varphi^{-1})}{\partial u^i}(\varphi(p)) + f(p)\frac{\partial(g \circ \varphi^{-1})}{\partial u^i}(\varphi(p)) \\
&= g(p)\frac{\partial f}{\partial x^i}(p) + f(p)\frac{\partial g}{\partial x^i}(p) \qquad \square
\end{aligned}$$

Example

$$\frac{\partial x^j}{\partial x^i}(p) = \delta_i^j \quad \forall 1 \leq i \leq n,\ 1 \leq j \leq n,$$

follows from $(x^j \circ \varphi^{-1})(u^1, \dots, u^n) = u^j$.

It turns out that **any** tangent vector can be written as a linear combination of the $\frac{\partial}{\partial x^i}\Big|_p$'s in a **unique** way:

Proposition

For all $p \in M$ and any local chart $(\varphi = (x^1, \ldots, x^n), U)$ with $p \in U$, the set of tangent vectors $\left\{ \frac{\partial}{\partial x^i}\Big|_p ,\ 1 \leq i \leq n \right\}$ is **basis** of T_pM.

Proof:

Linear independence of the $\frac{\partial}{\partial x^i}\Big|_p$:

- assume $\exists\ (c^1, \ldots, c^n) \in \mathbb{R}^n \neq 0$, such that
$$v_0 := \sum_{i=1}^{n} c^i \left. \frac{\partial}{\partial x^i} \right|_p = 0$$
as a linear map
- then $\exists$ at least one $1 \leq j \leq n$, such that $c^j \neq 0$, and we obtain
$$v_0(x^j) = c^j \neq 0$$
which is a contradiction to $v_0 = 0$

$T_pM = \mathrm{span}_{\mathbb{R}} \left\{ \frac{\partial}{\partial x^i}\Big|_p ,\ 1 \leq i \leq n \right\}$:

- assume wlog $\varphi(U) = B_r(0)$ [after possibly shrinking U and translating φ via constant vector $-\varphi(p) \in \mathbb{R}^n$]
- for any $g \in C^\infty(\varphi(U))$ obtain using the **fundamental theorem of calculus** and

$$g_j(q) := \int_0^1 \frac{\partial g}{\partial u^j}(tq)dt \quad \forall\ q \in \varphi(U),\ 1 \leq j \leq n,$$

  the identity

$$g = g(0) + \sum_{j=1}^{n} g_j u^j$$

- $\forall\ f \in C^\infty(U)$ get via $g := f \circ \varphi^{-1}$

$$f = g \circ \varphi = f(p) + \sum_{j=1}^{n} f_j x^j$$

(continuation of proof)

- acting with $\frac{\partial}{\partial x^i}\big|_p$ on both sides of above eqn. yields

$$f_i(p) = \frac{\partial f}{\partial x^i}(p) \quad \forall\ 1 \leq i \leq n$$

- for any $v \in T_pM$ fixed get by using $x^i(p) = 0$ for all $1 \leq i \leq n$ and the Leibniz rule

$$v(f) = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(p) v(x^i)$$

- hence

$$v = \sum_{i=1}^{n} v(x^i)\ \frac{\partial}{\partial x^i}\bigg|_p ,$$

meaning that $\left\{ \frac{\partial}{\partial x^i}\big|_p,\ 1 \leq i \leq n \right\}$ spans T_pM as claimed

□

Immediate consequence:

Corollary

The **dimensions** of a smooth manifold M and its tangent space T_pM **coincide** for all $p \in M$.

Question: How do the tangent vectors of the form $\sum_i c^i \left.\frac{\partial}{\partial x^i}\right|_p$ behave under a change of coordinates?

$\rightsquigarrow$ to answer this, we need **differentials** of smooth maps

Definition

Let M be a smooth manifold of dimension m and N be a smooth manifold of dimension n.

- The **differential at a point $p \in M$ of a smooth function** $f \in C^\infty(M)$ is defined as the linear map

$$df_p : T_pM \to \mathbb{R}, \quad v \mapsto v(f).$$

  In a given local coordinate system $\varphi = (x^1, \dots, x^m)$ on M that covers p, df_p is of the form

$$df_p : \left.\frac{\partial}{\partial x^i}\right|_p \mapsto \frac{\partial f}{\partial x^i}(p).$$

- (continued on next page)

Definition (continuation)

- The **differential at a point** $p \in M$ **of a smooth map** $F : M \to N$ in given local coordinate systems $\varphi = (x^1, \dots, x^m)$ on M and $\psi = (y^1, \dots, y^n)$ on N covering $p \in M$ and $F(p) \in N$, respectively, is defined as the linear map

$$dF_p : T_pM \to T_{F(p)}N, \quad \frac{\partial}{\partial x^i}\bigg|_p \mapsto \sum_{j=1}^{n} \frac{\partial F^j}{\partial x^i}(p) \, \frac{\partial}{\partial y^j}\bigg|_{F(p)},$$

where we have used the notation $F^j := y^j \circ F$. The **rank** of F is the rank of the linear map $dF_p : T_pM \to T_{F(p)}N$, which coincides with the rank of the **Jacobi matrix** of F at p in the local coordinate systems φ, ψ,

$$\left(\frac{\partial F^j}{\partial x^i}(p) \right)_{ji} \in \mathrm{Mat}(n \times m, \mathbb{R}).$$

[j = row, i = column **]**

Similar to real analysis, differentials fulfils a **chain rule**:

Lemma

Let M, N, P be smooth manifolds and $F : M \to N$, $G : N \to P$, smooth maps. Then

$$d(G \circ F)_p = dG_{F(p)} \circ dF_p$$

for all $p \in M$.

Proof: For any $v \in T_pM$ and $f \in C^\infty(P)$ we have

$$\begin{aligned} d(G \circ F)_p(v)(f) &= v(f \circ G \circ F) \\ &= dF_p(v)(f \circ G) = dG_{F(p)}(dF_p(v))(f). \end{aligned}$$

□

Example

For any local coordinate system $\varphi = (x^1, \dots, x^n)$ on M covering $p \in M$, $d\varphi_p : T_pM \to T_{\varphi(p)}\mathbb{R}^n \cong \mathbb{R}^n$ is of the form

$$d\varphi_p = (dx^1, \dots, dx^n)_p = (dx_p^1, \dots, dx_p^n),$$

$$dx_p^j \left(\left. \frac{\partial}{\partial x^i} \right|_p \right) = \delta_i^j.$$

Viewed as $n \times n$-matrix **[**by identifying $d\varphi_p$ with its Jacobi matrix**]**, $d\varphi_p = \mathbb{1}_n$.

Lemma

For two local coordinate systems $\varphi = (x^1, \ldots, x^n)$ and $\psi = (y^1, \ldots, y^n)$ with overlapping domain on M we obtain for all p contained in the overlap the identity

$$\left.\frac{\partial}{\partial x^i}\right|_p = \sum_{j=1}^n \frac{\partial y^j}{\partial x^i}(p) \left.\frac{\partial}{\partial y^j}\right|_p .$$

Proof:

For any $f \in C^\infty(M)$, $1 \leq i \leq n$, p in the overlap we calculate

$$\begin{aligned}
\frac{\partial f}{\partial x^i}(p) &= \frac{\partial (f \circ \varphi^{-1})}{\partial u^i}(\varphi(p)) = \frac{\partial (f \circ \psi^{-1} \circ \psi \circ \varphi^{-1})}{\partial u^i}(\varphi(p)) \\
&= \sum_{j=1}^n \frac{\partial (f \circ \psi^{-1})}{\partial u^j}(\psi(p)) \frac{\partial (u^j \circ \psi \circ \varphi^{-1})}{\partial u^i}(\varphi(p)) \\
&= \sum_{j=1}^n \frac{\partial f}{\partial y^j}(p) \frac{\partial y^j}{\partial x^i}(p).
\end{aligned}$$

□

Properties of differentials are used to classify smooth maps as follows:

Definition

- A smooth map between smooth manifolds $F : M \to N$ is called an **immersion** if $dF_p : T_pM \to T_{F(p)}N$ is injective for all $p \in M$.
- $F : M \to N$ is called a **submersion** if $dF_p : T_pM \to T_{F(p)}N$ is surjective for all $p \in M$.
- An immersion $F : M \to N$ is called an **embedding** if F is injective and an homeomorphism onto its image $F(M) \subset N$ equipped with the subspace topology.
- A smooth map $F : M \to N$ between smooth manifolds of the same dimension is called a **local diffeomorphism** if for all $p \in M$ there exists an open neighbourhood of p, $U \subset M$, such that $F|_U : U \to N$ is a diffeomorphism onto its image.

END OF LECTURE 4

Next lecture:

- smooth manifold analogue to inverse function theorem
- more examples of smooth maps
- submanifolds of smooth manifolds