Differential geometry
Lecture 3: Tangent spaces (part 1)

David Lindemann

University of Hamburg
Department of Mathematics
Analysis and Differential Geometry & RTG 1670

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1. The vector space of smooth functions

2. Tangent vectors on $\mathbb{R}^n$

3. Tangent vectors on smooth manifolds

Recap of lecture 2:

- studied the **implicit/inverse function theorem**
- defined **smooth submanifolds of** $\mathbb{R}^n$, proved that they are **smooth manifolds**
- the **Cartesian product** of smooth manifolds is a smooth manifold
- defined what **smooth maps between manifolds** are
- erratum: The graph of $f = x^2 + y^2 - 1$ is **not a hyperboloid**, but, up to translation by $(0, 0, -1)$, a **paraboloid**.

Definition

The **vector space of smooth functions** $C^\infty(M)$ on a smooth manifold M consists of all smooth functions $f : M \to \mathbb{R}$. For $U \subset M$ open, the set

$$C^\infty(U) := \{f : U \to \mathbb{R} \text{ smooth}\}$$

is also a vector space. Elements of $C^\infty(U)$ are called **local smooth functions** (w.r.t. U).

- $C^\infty(M)$ is an commutative ring with unit the constant function $f \equiv 1$
- same true for $C^\infty(U)$
- the linear map $C^\infty(M) \ni f \mapsto f|_U \in C^\infty(U)$ is called the **restriction map** w.r.t. $U \subset M$
- for $M \setminus U$ having nonempty interior, the restriction map is **neither** surjective **nor** injective

Example

Let $(x^1, \dots, x^n)$ be local coordinates on $U \subset M$. Then the **coordinate functions** $x^i : U \to \mathbb{R}$, $1 \le i \le n$, are (local) smooth functions.

We will make use of the following special type of functions:

Definition

Let M be a smooth manifold, $U \subset M$ open, and $V \subset U$ compactly embedded with nonempty interior. A **bump function** w.r.t. that given data is a compactly supported smooth function $b \in C^\infty(M)$, such that

$$b|_{\overline{V}} \equiv 1, \quad \operatorname{supp}(b) \subset U.$$

- for any permissible data V, U, M as above there exists a bump function w.r.t. that data
- commonly bump functions are defined as **any** compactly supported smooth functions on M, also called **test functions**

Using bump functions, we can extend local smooth functions to globally defined smooth functions:

Definition

Let $U \subset M$ open and $f \in C^\infty(U)$. Let $V \subset U$ be compactly embedded with nonempty interior and $b \in C^\infty(M)$ be a bump function w.r.t. V, U, M. Then

$$(bf)(p) := \begin{cases} b(p)f(p), & p \in U, \\ 0, & p \in M \setminus U \end{cases}$$

is called the **trivial extension** of $b|_U f \in C^\infty(U)$ to M and is contained in $C^\infty(M)$.

- if $f \in C^\infty(U)$, $g \in C^\infty(U')$, $U, U' \in M$ open with nonempty intersection, and $f|_W = g|_W$ for some open $W \subset U \cap U'$, then there exists a bump function $b \in C^\infty(M)$ w.r.t. fitting data, such that

$$bf = bg \in C^\infty(M)$$

- proof of above claim needs **locally finite partitions of unity**

Before defining what tangent vectors should be on smooth manifolds, recall:

Definition

A **tangent vector** at $p \in \mathbb{R}^n$ is an equivalence class of smooth curves through p, $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$, $\gamma(0) = p$, where

$$[\gamma] = [\widetilde{\gamma}] \quad :\Leftrightarrow \quad \gamma'(0) = \widetilde{\gamma}'(0).$$

Tangent vectors at p act on local smooth functions $f \in C^\infty(U)$, $U \subset \mathbb{R}^n$ open, $p \in U$, via

$$[\gamma]f := \left.\frac{d(f \circ \gamma)}{dt}\right|_{t=0},$$

which is the **directional derivative** of f at p in direction $\gamma'(0)$.

- the set of tangent vectors at $p \in \mathbb{R}^n$ is a real vector space and called **tangent space of $\mathbb{R}^n$ at p**
- the disjoint union of all tangent spaces at p is called the **tangent space of $\mathbb{R}^n$**, it is (as a set) isomorphic to $\mathbb{R}^n \times \mathbb{R}^n$ ("base-points & vectors")

We also know when tangent vectors are tangential to a smooth submanifold of $\mathbb{R}^n$:

Remark

A tangent vector $[\gamma]$ at p is called tangential to a smooth $m < n$-dimensional submanifold M of $\mathbb{R}^n$ if for any locally defining function $F : M \cap U \to \mathbb{R}^n$

$$dF_p \cdot \gamma'(0) \in \mathbb{R}^m \times \{0\}.$$

or, equivalently, if $\exists$ open nbh. $U \subset \mathbb{R}^n$ of p, such that

$$df_p \cdot \gamma'(0) = 0$$

for a smooth map $f : U \to \mathbb{R}^{n-m}$ with Jacobi matrix of maximal rank with $M \cap U = \{x \in U \mid f(x) = 0\}$.

Problem 1: How can we generalize tangent vectors (and the tangent space) of $\mathbb{R}^n$ to general smooth manifolds?
Problem 2: What is a good choice for the topology of the tangent space? What additional structure does it have?
$\rightsquigarrow$ use how tangent vectors act on functions:

Definition

Let M be a smooth manifold. A **tangent vector** v **at** $p \in M$ is a linear map

$$v : C^\infty(M) \to \mathbb{R},$$

that fulfils the **Leibniz rule**

$$v(fg) = g(p)v(f) + f(p)v(g).$$

The **tangent space at** $p \in M$

$$T_pM := \{v : C^\infty(M) \to \mathbb{R} \mid v \text{ tangent vector at p}\}$$

is the real vector space of all tangent vectors v at $p \in M$.

One calls tangent vectors **local** (or **local objects**) because of the following result:

Proposition

Let $f, g \in C^\infty(M)$ and assume $\exists\ U \subset M$ open, such that $f|_U = g|_U$. Then for all $p \in U$ and all $v \in T_pM$

$$v(f) = v(g).$$

Furthermore, if f is locally constant near $p \in M$, then

$$v(f) = 0 \quad \forall v \in T_pM.$$

Sketch of proof:

- by *linearity* of $v \in T_pM$: for first point suffices to show that
$$v(f) = 0$$
for all f locally vanishing on $U \subset M$ open near p
- (continued on next page)

Sketch of proof (continuation):

- choose V open and compactly embedded in U with nonempty interior, $p \in V$, and bump function $b \in C^\infty(M)$ w.r.t. V, U, M, get with Leibniz rule

$$0 = v(0) = v(bf) = f(p)v(b) + b(p)v(f) = 0 + v(f)$$

- second point: use first point to get $v(f) = v(c)$ if $f|_U \equiv c \in \mathbb{R}$, get

$$v(f) = v(c) = cv(1) = cv(1\cdot 1) = c(1\cdot v(1)+1\cdot v(1)) = 2cv(1) = 2v(f),$$

hence $v(f) = 0$

Remark

The latter proposition in particular implies that we can identify T_pU and T_pM for all $U \subset M$ open with $p \in M$. More precisely, $v \in T_pM$ defines $\widetilde{v} \in T_pU$ via any trivial extension $bf \in C^\infty(M)$ of $f \in C^\infty(U)$ and

$$\widetilde{v}(f) := v(bf),$$

which does not depend on the trivial extension as long b is w.r.t. V, U, M with p in the interior of V. On the other hand, $w \in T_pU$ defines $\widetilde{w} \in T_pM$ via

$$\widetilde{w}(f) := w(f|_U).$$

These two constructions are **inverse** to each other.

$\rightsquigarrow$ one can define tangent vectors $v \in T_pM$ as linear maps from the **germ of smooth functions** $\mathcal{F}_p$ **at** $p \in M$ to $\mathbb{R}$,

$$v : \mathcal{F}_p \to \mathbb{R}$$

so that v fulfils the Leibniz rule w.r.t. the product on $\mathcal{F}_p$. $\mathcal{F}_p$ is given by

$$\mathcal{F}_p := \{f \in C^\infty(U) \mid U \text{ open nbh. of } p \in M\}/_\sim,$$

where $f \sim g$ if $\exists$ open nbh. U of p contained in domains of f and g, such that $f|_U = g|_U$.

END OF LECTURE 3

Next lecture:

- tangent spaces [second part], in particular how to actually write down and calculate with tangent vectors