Differential geometry
Lecture 3: Tangent spaces (part 1)

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1 The vector space of smooth functions

2 Tangent vectors on $\mathbb{R}^n$

3 Tangent vectors on smooth manifolds
Recap of lecture 2:

- studied the implicit/inverse function theorem
- defined smooth submanifolds of $\mathbb{R}^n$, proved that they are smooth manifolds
- the Cartesian product of smooth manifolds is a smooth manifold
- defined what smooth maps between manifolds are
- erratum: The graph of $f = x^2 + y^2 - 1$ is not a hyperboloid, but, up to translation by $(0, 0, -1)$, a paraboloid.
The vector space of smooth functions $C^\infty(M)$ on a smooth manifold $M$ consists of all smooth functions $f : M \to \mathbb{R}$. For $U \subset M$ open, the set

$$C^\infty(U) := \{ f : U \to \mathbb{R} \text{ smooth} \}$$

is also a vector space. Elements of $C^\infty(U)$ are called local smooth functions (w.r.t. $U$).

- $C^\infty(M)$ is an commutative ring with unit the constant function $f \equiv 1$
- same true for $C^\infty(U)$
- the linear map $C^\infty(M) \ni f \mapsto f|_U \in C^\infty(U)$ is called the restriction map w.r.t. $U \subset M$
- for $M \setminus U$ having nonempty interior, the restriction map is neither surjective nor injective
Example

Let \((x^1, \ldots, x^n)\) be local coordinates on \(U \subset M\). Then the coordinate functions \(x^i : U \to \mathbb{R}, 1 \leq i \leq n\), are (local) smooth functions.

We will make use of the following special type of functions:

Definition

Let \(M\) be a smooth manifold, \(U \subset M\) open, and \(V \subset U\) compactly embedded with nonempty interior. A bump function w.r.t. that given data is a compactly supported smooth function \(b \in C^\infty(M)\), such that

\[b|_V \equiv 1, \quad \text{supp}(b) \subset U.\]

- for any permissible data \(V, U, M\) as above there exists a bump function w.r.t. that data
- commonly bump functions are defined as any compactly supported smooth functions on \(M\), also called test functions
Using bump functions, we can extend local smooth functions to globally defined smooth functions:

**Definition**

Let $U \subset M$ open and $f \in C^\infty(U)$. Let $V \subset U$ be compactly embedded with nonempty interior and $b \in C^\infty(M)$ be a bump function w.r.t. $V, U, M$. Then

$$(bf)(p) := \begin{cases} 
    b(p)f(p), & p \in U, \\
    0, & p \in M \setminus U
\end{cases}$$

is called the **trivial extension** of $b|_U f \in C^\infty(U)$ to $M$ and is contained in $C^\infty(M)$.

- if $f \in C^\infty(U), g \in C^\infty(U'), U, U' \in M$ open with nonempty intersection, and $f|_W = g|_W$ for some open $W \subset U \cap U'$, then there exists a bump function $b \in C^\infty(M)$ w.r.t. fitting data, such that

  $$bf = bg \in C^\infty(M)$$

- proof of above claim needs **locally finite partitions of unity**
Before defining what tangent vectors should be on smooth manifolds, recall:

**Definition**

A **tangent vector** at \( p \in \mathbb{R}^n \) is an equivalence class of smooth curves through \( p \), \( \gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^n \), \( \gamma(0) = p \), where

\[
[\gamma] = [\tilde{\gamma}] \iff \gamma'(0) = \tilde{\gamma}'(0).
\]

Tangent vectors at \( p \) act on local smooth functions \( f \in C^\infty(U), \ U \subset \mathbb{R}^n \) open, \( p \in U \), via

\[
[\gamma]f := \left. \frac{d(f \circ \gamma)}{dt} \right|_{t=0},
\]

which is the **directional derivative** of \( f \) at \( p \) in direction \( \gamma'(0) \).

- the set of tangent vectors at \( p \in \mathbb{R}^n \) is a real vector space and called **tangent space of** \( \mathbb{R}^n \) **at** \( p \)
- the disjoint union of all tangent spaces at \( p \) is called the **tangent space of** \( \mathbb{R}^n \), it is (as a set) isomorphic to \( \mathbb{R}^n \times \mathbb{R}^n \) ("base-points & vectors")
We also know when tangent vectors are tangential to a smooth submanifold of $\mathbb{R}^n$:

**Remark**

A tangent vector $[\gamma]$ at $p$ is called tangential to a smooth $m < n$-dimensional submanifold $M$ of $\mathbb{R}^n$ if for any locally defining function $F : M \cap U \to \mathbb{R}^n$

$$dF_p \cdot \gamma'(0) \in \mathbb{R}^m \times \{0\}.$$ 

or, equivalently, if $\exists$ open nbh. $U \subset \mathbb{R}^n$ of $p$, such that

$$df_p \cdot \gamma'(0) = 0$$

for a smooth map $f : U \to \mathbb{R}^{n-m}$ with Jacobi matrix of maximal rank with $M \cap U = \{x \in U \mid f(x) = 0\}$. 

Problem 1: How can we generalize tangent vectors (and the tangent space) of $\mathbb{R}^n$ to general smooth manifolds?

Problem 2: What is a good choice for the topology of the tangent space? What additional structure does it have?

\[\text{\rightsquigarrow use how tangent vectors act on functions:}\]

**Definition**

Let $M$ be a smooth manifold. A **tangent vector** $v$ at $p \in M$ is a linear map

\[ v : C^\infty(M) \rightarrow \mathbb{R}, \]

that fulfils the **Leibniz rule**

\[ v(fg) = g(p)v(f) + f(p)v(g). \]

The **tangent space at** $p \in M$

\[ T_p M := \{ v : C^\infty(M) \rightarrow \mathbb{R} \mid v \text{ tangent vector at } p \} \]

is the real vector space of all tangent vectors $v$ at $p \in M$. 
One calls tangent vectors **local** (or **local objects**) because of the following result:

**Proposition**

Let \( f, g \in C^\infty(M) \) and assume \( \exists \ U \subset M \) open, such that \( f|_U = g|_U \). Then for all \( p \in U \) and all \( v \in T_pM \)

\[
  v(f) = v(g).
\]

Furthermore, if \( f \) is locally constant near \( p \in M \), then

\[
  v(f) = 0 \quad \forall v \in T_pM.
\]

**Sketch of proof:**

- by linearity of \( v \in T_pM \): for first point suffices to show that
  \[
  v(f) = 0
  \]
  for all \( f \) locally vanishing on \( U \subset M \) open near \( p \)

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Sketch of proof (continuation):

- choose $V$ open and compactly embedded in $U$ with nonempty interior, $p \in V$, and bump function $b \in C^\infty(M)$ w.r.t. $V, U, M$, get with Leibniz rule

$$
0 = v(0) = v(bf) = f(p)v(b) + b(p)v(f) = 0 + v(f)
$$

- second point: use first point to get $v(f) = v(c)$ if $f|_U \equiv c \in \mathbb{R}$, get

$$
v(f) = v(c) = cv(1) = c(1\cdot v(1)) = 2cv(1) = 2v(f), \quad \text{hence } v(f) = 0
$$
Remark

The latter proposition in particular implies that we can identify $T_p U$ and $T_p M$ for all $U \subset M$ open with $p \in M$. More precisely, $v \in T_p M$ defines $\tilde{v} \in T_p U$ via any trivial extension $b f \in C^\infty(M)$ of $f \in C^\infty(U)$ and

$$\tilde{v}(f) := v(b f),$$

which does not depend on the trivial extension as long $b$ is w.r.t. $V$, $U$, $M$ with $p$ in the interior of $V$. On the other hand, $w \in T_p U$ defines $\tilde{w} \in T_p M$ via

$$\tilde{w}(f) := w(f|_U).$$

These two constructions are inverse to each other.
one can define tangent vectors $v \in T_p M$ as linear maps from the germ of smooth functions $\mathcal{F}_p$ at $p \in M$ to $\mathbb{R}$,

$$v : \mathcal{F}_p \rightarrow \mathbb{R}$$

so that $v$ fulfils the Leibniz rule w.r.t. the product on $\mathcal{F}_p$. $\mathcal{F}_p$ is given by

$$\mathcal{F}_p := \{ f \in C^\infty(U) \mid U \text{ open nbh. of } p \in M \}/\sim,$$

where $f \sim g$ if $\exists$ open nbh. $U$ of $p$ contained in domains of $f$ and $g$, such that $f|_U = g|_U$. 
END OF LECTURE 3

Next lecture:

- tangent spaces [second part], in particular how to actually write down and calculate with tangent vectors