Differential geometry
Lecture 2: Smooth maps and the IFT

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Recap of lecture 1:

- revisited some **topology**
- defined (maximal) **atlases**
- defined structure of a **smooth manifold**
- have seen that a given atlas specifies max. atlas **uniquely**
- erratum: charts $\varphi : U \to \varphi(U)$ are not just *any* maps, but **homeomorphisms**

Before stating the implicit function theorem, recall the following:

Inverse function theorem

Let $U \subset \mathbb{R}^n$ be open, $F : U \to \mathbb{R}^n$ be smooth, and assume that the Jacobi matrix $dF|_p$ is **invertible** for some $p \in \mathbb{R}^n$. Then $\exists$ open nbhs. $V \subset \mathbb{R}^n$ of p and $W \subset \mathbb{R}^n$ of $F(p)$, such that

$$F : V \to W$$

is invertible and its inverse $F^{-1} : W \to V$ is smooth.

- F is **not** necessarily globally invertible
- there are version of the inverse function theorem for e.g. analytic maps, holomorphic functions, Frechét-differentiable maps between Banach spaces, and of course for smooth maps between smooth manifolds

Implicit function theorem (smooth version)

Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$, $(x, y) \mapsto f(x, y)$, be smooth, $f(p) = 0$ for $p = (x_0, y_0) \in \mathbb{R}^n \times \mathbb{R}^m$, and assume that that the Jacobi matrix of f with respect to y at p,

$$d_y f|_p = \begin{pmatrix} \frac{df_1}{dy^1}(p) & \dots & \frac{df_1}{dy^m}(p) \\ \vdots & \ddots & \vdots \\ \frac{df_m}{dy^1}(p) & \dots & \frac{df_m}{dy^m}(p) \end{pmatrix},$$

is invertible. Then $\exists$ open nbhs. $U \subset \mathbb{R}^n$ of x_0 and $V \subset \mathbb{R}^m$ of y_0, s.t. $\exists$ a **unique** smooth map $g : U \to V$ fulfilling

$$f(x, y) = 0,\ x \in U,\ y \in V \quad \Leftrightarrow \quad y = g(x),$$

in particular we have $g(x_0) = y_0$.

- there are other versions of the IFT, e.g. if f is analytic g will be analytic, or a version of the IFT for Banach spaces
- the IFT follows from the inverse function theorem

Important examples of smooth manifolds are the following:

Definition

An $m < n$-**dimensional smooth submanifold of** $\mathbb{R}^n$ is a subset $M \subset \mathbb{R}^n$, such that $\forall\, p \in M\ \exists$ open nbh. $U \subset \mathbb{R}^n$ of p and a smooth map $f : U \to \mathbb{R}^{n-m}$ with Jacobi matrix of maximal rank $n - m$ for all points in U fulfilling

$$M \cap U = \{x \in U \mid f(x) = 0\}.$$

Question: Are smooth submanifolds of $\mathbb{R}^n$ actually smooth manifolds?

Answer: Yes! $\rightsquigarrow$ use the IFT:

Lemma

m-dimensional smooth submanifolds M of $\mathbb{R}^n$ can be **locally** written as graphs of smooth maps

$$g : V \to \mathbb{R}^{n-m}, \quad V \subset \mathbb{R}^m \text{ open}.$$

$\rightsquigarrow$ still need to find **atlas** on M:

Corollary

For any point p in an m-dim. smooth submfd. $M \subset \mathbb{R}^n$ $\exists$ (after possibly reordering coordinates on $\mathbb{R}^n$) open nbh. $U \subset \mathbb{R}^n$ of p and a smooth invertible map with smooth inverse

$$F : U \to \mathbb{R}^n$$

such that

$$F|_{U \cap M} : (x^1, \dots, x^n) \mapsto (x^1, \dots, x^m, 0, \dots, 0).$$

Get **local coordinates** via:

- cover M with open sets U as above
- get charts on M:

$$(\varphi = \mathrm{pr}_{\mathbb{R}^m} \circ F|_{U \cap M}, U \cap M)$$

Yet another way to find examples of smooth manifolds is by taking the product of two manifolds:

Lemma

Let M with atlas $\mathcal{A}$ and N with atlas $\mathcal{B}$ be smooth manifolds. Then the **Cartesian product** of M and N,

$$M \times N,$$

together with the **product atlas**

$$\mathcal{A} \times \mathcal{B} := \{(\varphi \times \psi, U \times V) \mid (\varphi, U) \in \mathcal{A}, (\psi, V) \in \mathcal{B}\}$$

is a smooth manifold.

Proof: Exercise.
Remark: The product atlas is in general not maximal.
Examples:

- $S^1 \times S^1 = T^2$, the 2-torus
- $S^1 \times \mathbb{R}$, the cylinder
- $\mathbb{R}^m \times \mathbb{R}^n \cong \mathbb{R}^{m+n}$

Problem: What does "$\cong$" mean?

Maps between manifolds are called smooth if they are smooth in local coordinates:

Definition

Let M and N be smooth manifolds. A continuous map $f : M \to N$ is called **smooth** if for all charts (φ, U) of M, (ψ, V) of N,

$$\psi \circ f \circ \varphi^{-1} : \varphi(U \cap f^{-1}(V)) \to \psi(V)$$

is smooth. If f is invertible and its inverse $f^{-1} : N \to M$ is smooth, f is called a **diffeomorphism**.

- a map $f : U \to V$ between open sets U of $\mathbb{R}^n$ and V of $\mathbb{R}^m$ is smooth in the sense of real analysis if and only if it is smooth in the sense above where U and V are viewed as smooth manifolds equipped with the restriction of the canonical coordinates
- the set of diffeomorphisms on an $n \geq 1$-dim. smooth manifold M, $\mathrm{Diff}(M)$, forms an **infinite dimensional Lie group**

END OF LECTURE 2

Next lecture:

- tangent spaces [first part]