Differential geometry
Lecture 1: Smooth manifolds

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Definition

Let M be a set. A **topology** $\boldsymbol{\tau}$ on M is a collection of subsets of M, such that

- M and $\emptyset$ are contained in $\boldsymbol{\tau}$
- arbitrary unions of sets in $\boldsymbol{\tau}$ are in $\boldsymbol{\tau}$
- finite intersections of sets in $\boldsymbol{\tau}$ are in $\boldsymbol{\tau}$

$(M, \boldsymbol{\tau})$ is called **topological space** and sets in $\boldsymbol{\tau}$ are called **open**.

Examples:

- $\mathbb{R}^n$ with topology induced by any norm [most important example for this course]
- $(M, \boldsymbol{\tau})$ with $\boldsymbol{\tau} = \{\emptyset, M\}$ for any non-empty set M
- $(M, \boldsymbol{\tau})$ with $\boldsymbol{\tau} = \{U \subset M\}$ for any non-empty set M [all subsets of M are open]

Definition

A **basis of the topology** of $(M, \boldsymbol{\tau})$ is a collection of open sets $\mathcal{B}$, such that for all open sets U $\exists$ index set I and corresponding $B_i \in \mathcal{B}$ with

$$\bigcup_{i \in I} B_i = U.$$

This means: A basis of a topology **generates** the topology.
Examples:

- open balls $B_r(p) \subset \mathbb{R}^n$ w.r.t. any norm, $r \in \mathbb{Q}_{>0}$, $p \in \mathbb{Q}^n$
- any topology is a basis of itself

Definition

- $(M, \boldsymbol{\tau})$ is called **Hausdorff** if $\forall p \neq q \in M$ $\exists$ U, V open with $p \in U$, $q \in V$, such that $U \cap V = \emptyset$.
- $(M, \boldsymbol{\tau})$ is called **second countable** if its topology has a countable basis.

Next define central objects we will study in this course:

Definition

Let M be a second countable Hausdorff topological space. An n-**dimensional smooth atlas on** M is a collection of maps

$$\mathcal{A} = \{(\varphi_i, U_i) \mid i \in A\}, \quad \varphi_i : U_i \to \varphi_i(U_i) \subset \mathbb{R}^n,$$

such that all $U_i \subset M$ are open, all φ_i are homeomorphisms, and

- $\{U_i, i \in I\}$ is an open covering of M
- $\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$ are smooth for all $i, j \in A$

- (φ_i, U_i), $i \in A$, are called **charts** on M
- $\varphi_i \circ \varphi_j^{-1}$, whenever defined, are called **transition functions**
- from here on: atlas $=$ n-dimensional smooth atlas

Definition

Two atlases $\mathcal{A}$ and $\mathcal{B}$ on M are called **equivalent** if $\mathcal{A} \cup \mathcal{B}$ is an atlas on M. [Notation: $[\mathcal{A}] = [\mathcal{B}]$]

Definition

An atlas $\mathcal{A}$ on M is called **maximal** if for all atlases $\mathcal{B}$ on M with $[\mathcal{A}] = [\mathcal{B}]$ we have

$$\mathcal{B} \subset \mathcal{A}.$$

- atlases form a **partially ordered set** w.r.t. "$\subset$"
- $\rightsquigarrow$ can use Zorn's lemma

Theorem A

Every atlas is contained in a maximal atlas. If $\mathcal{A}_1$ and $\mathcal{A}_2$ are two maximal atlases on M, such that there exists an atlas $\mathcal{B}$ on M with

$$[\mathcal{A}_1] = [\mathcal{B}] \text{ and } [\mathcal{A}_2] = [\mathcal{B}]$$

then $\mathcal{A}_1$ and $\mathcal{A}_2$ already coincide.

Sketch of proof:

- The first point follows by using Zorn's lemma.

- The second point follows by writing

$$\varphi \circ \psi^{-1} = (\varphi \circ \phi^{-1}) \circ (\phi \circ \psi^{-1})$$

for the transition functions of charts with overlaps φ in $\mathcal{A}_1$, $\psi \in \mathcal{A}_2$, $\phi \in \mathcal{B}$. Since $\mathcal{B}$ is an atlas, its charts cover M and the claim follows.

Remark

Theorem A implies that, in order to **uniquely** specify a maximal atlas on M it suffices to specify **any** atlas on M, e.g. a finite atlas.

Now we have all tools at hand to define **smooth manifolds**:

Definition

A second countable Hausdorff space M together with an n-dimensional maximal atlas is called **smooth manifold** of dimension $\dim(M) = n$.

An immediate consequence of the second countability property is:

Proposition

Every maximal atlas contains a countable atlas.

Proof: Exercise!
Examples of smooth manifolds:

Examples

- $\mathbb{R}^n$ with atlas containing only one chart $(\mathrm{id} = (u^1, \dots, u^n), \mathbb{R}^n)$. The maps

$$u^i : p = (p_1, \dots, p_n) \mapsto p_i$$

  are called **canonical coordinates** on $\mathbb{R}^n$.
- The n-sphere $S^n \subset \mathbb{R}^{n+1}$ with atlas containing the two charts $(\sigma_+, S^n \setminus \{p_+\})$ and $(\sigma_-, S^n \setminus \{p_-\})$.

Examples (continuation)

- $\mathbb{R}P^n = \left\{[x^1 : \ldots : x^{n+1}] \;\middle|\; x = (x^1, \ldots, x^{n+1}) \in \mathbb{R}^{n+1} \setminus \{0\}\right\}$, the real projective n-space, with $n+1$ charts

$$\varphi_i : \pi\left(\mathbb{R}^{n+1} \setminus \{x^i = 0\}\right) \to \mathbb{R}^n,$$
$$[x^1 : \ldots : x^{i-1} : x^i : x^{i+1} : \ldots : x^{n+1}]$$
$$\mapsto \left(\frac{x^1}{x^i} : \ldots : \frac{x^{i-1}}{x^i} : \widehat{x^i} : \frac{x^{i+1}}{x^i} : \ldots : \frac{x^{n+1}}{x^i}\right),$$

  called **homogeneous coordinates** on $\mathbb{R}P^n$
- the graph of a smooth function $f : U \to \mathbb{R}$, $U \subset \mathbb{R}^n$ open,

$$\mathrm{graph}(f) = \{(x, f(x)) \mid x \in U\} \subset \mathbb{R}^{n+1}$$

  with one chart $\varphi : \mathrm{graph}(f) \to U$, $(x, f(x)) \mapsto x$
- any open subset $U \subset M$ of a smooth manifold with atlas given by restricting an atlas on M to U

Similar to the term "coordinates" on $\mathbb{R}^n$ we call the following objects local coordinates on smooth manifolds:

Definition

Let $(u^1, \dots, u^n)$ denote the canonical coordinates on $\mathbb{R}^n$. For any chart (φ, U) on an n-dim. smooth manifold M, we have

$$\varphi = (u^1 \circ \varphi, \dots, u^n \circ \varphi) =: (x^1, \dots, x^n).$$

The smooth functions x^i are called **local coordinate functions**, and $\varphi = (x^1, \dots, x^n)$ is called **local coordinate system** (or simply **local coordinates**) on M

END OF LECTURE 1

Next lecture:

- implicit function theorem and submanifolds of $\mathbb{R}^n$
- smooth maps and diffeomorphisms
- methods to construct smooth manifolds