Differential geometry
Lecture 19: Geodesics & curvature of pseudo-Riemannian submanifolds

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Recap of lecture 18:

- defined **Riemann curvature tensor** R
- discussed **geometric interpretation** of R via parallel transport around infinitesimal loops
- proved several **identities of** R, e.g. the Bianchi-identities
- interpreted R as measure of how much **second covariant derivatives of vector fields** fail to commute
- studied **sectional curvature**, showed how to recover R from sectional curvatures
- defined **Ricci curvature** and **scalar curvature**

Notation: (M, g) is a **pseudo-Riemannian submanifold** of $(\overline{M}, \overline{g})$ if $M \subset \overline{M}$ and $g = \overline{g}|_{TM \times TM}$. $\overline{\cdot}$ does **not** denote the topological closure in this lecture.
Recall that for a pseudo-Riemannian submanifold (M, g) of $(\overline{M}, \overline{g})$ we have an **orthogonal splitting** $T\overline{M}|_M = TM \oplus TM^{\perp}$.

Definition

The metric $g = \overline{g}|_{TM \times TM}$ of a pseudo-Riemannian submanifold (M, g) of $(\overline{M}, \overline{g})$ is called **first fundamental form**.

Question: Except from the metric, what other geometrical structures (connection, curvature,...) are induced by the **ambient manifold** of a pseudo-Riemannian submanifold? How are they related to **intrinsic** geometrical structures (LCC connection, curvature,...) of the pseudo-Riemannian submanifold?
Answer: Surprisingly **nicely**!

Definition

Let (M, g) be a pseudo-Riemannian submanifold of $(\overline{M}, \overline{g})$. We **identify** $\mathfrak{X}(M)$ with tangential sections of $T\overline{M}|_M = TM \oplus TM^\perp \to M$. Sections in the subbundle $TM^\perp \to M$ are called **normal sections**, denoted by $\mathfrak{X}(M)^\perp$. We further define **tangential and normal projections** as

$$\tan : TM \oplus TM^\perp \to TM, \quad \mathrm{nor} : TM \oplus TM^\perp \to TM^\perp,$$

given **fibrewise** by $\tan(v + \xi) = v$ and $\mathrm{nor}(v + \xi) = \xi$ for all $v \in T_pM$ and all $\xi \in T_pM^\perp$.

The following holds for **all** smooth submanifolds.

Lemma A

Let $M \subset \overline{M}$ be a smooth submanifold. Let further $X, Y \in \mathfrak{X}(M)$ be arbitrary and $\overline{X}, \overline{Y} \in \mathfrak{X}(\overline{M})$ be arbitrary extensions of X, Y to $\overline{M}$, i.e. $\overline{X}_p = X_p$ and $\overline{Y}_p = Y_p$ for all $p \in M$. Then $[\overline{X}, \overline{Y}]_p \in T_pM$ for all $p \in M$.

Proof: Follows from the concept of ϕ-**related** vector fields and the compatibility with the Lie bracket. □

Next we will study how the Levi-Civita connection in an ambient pseudo-Riemannian manifold **defines connections in** $TM \to M$ **and** $TM^\perp \to M$ for any pseudo-Riemannian submanifold (M, g).

Lemma B

Let M be a pseudo-Riemannian submanifold of $(\overline{M}, \overline{g})$ and let $\overline{\nabla}$ denote the Levi-Civita connection of $(\overline{M}, \overline{g})$. Let $X \in \mathfrak{X}(M)$ and $Y \in \Gamma(T\overline{M}|_M)$ with arbitrary extensions $\overline{X}, \overline{Y} \in \mathfrak{X}(M)$. Then

$$\overline{\nabla}_{\overline{X}}\overline{Y}\Big|_M \in \Gamma(T\overline{M}|_M)$$

is **independent of the chosen extensions** $\overline{X}$ and $\overline{Y}$.

Proof: Follows from $\overline{\nabla}_{\overline{X}}\overline{Y}\Big|_p = \overline{\nabla}_{\gamma'}\overline{Y}_\gamma|_{t=0}$ for all $p \in M$ for any **integral curve** $\gamma : (-\varepsilon, \varepsilon) \to \overline{M}$ of $\overline{X}$ with initial condition $\gamma(0) = p$ and the fact that $\overline{X}$ restricts by assumption to a vector field on M. □

Corollary

The Levi-Civita connection of a pseudo-Riemannian manifold $(\overline{M}, \overline{g})$ with pseudo-Riemannian submanifold (M, g) **induces a connection** in $TM \oplus TM^{\perp} \to M$.

Problem: It is **not clear** at this point if the above connection restricts to the subbundles $TM \to M$ and $TM^{\perp} \to M$, respectively.

Solution 1:

Proposition A

Let (M, g) be a pseudo-Riemannian submanifold of $(\overline{M}, \overline{g})$. Then the **Levi-Civita connection ∇ of (M, g)** is precisely the **tangent part of the Levi-Civita connection $\overline{\nabla}$ of $(\overline{M}, \overline{g})$** restricted to $\mathfrak{X}(M) \times \mathfrak{X}(M)$, i.e.

$$\nabla_X Y = \tan \overline{\nabla}_X Y$$

for all $X, Y \in \mathfrak{X}(M)$.

Proof: (see next page)

(continuation of proof)

- observe that $\tan\overline{\nabla}$ **is a connection**, follows from fibrewise linearity of $\tan$
- $\rightsquigarrow$ remains to show that $\tan\overline{\nabla}$ is **metric** and **torsion-free**
- obtain for all $X, Y, Z \in \mathfrak{X}(M)$ with arbitrary respective extensions $\overline{X}, \overline{Y}, \overline{Z} \in \mathfrak{X}(\overline{M})$ for all $p \in M$

$$
\begin{aligned}
&(\tan\overline{\nabla}_X g)(Y, Z)|_p \\
&= \big(X(g(Y, Z)) - g(\tan\overline{\nabla}_X Y, Z) - g(Y, \tan\overline{\nabla}_X Z)\big)\big|_p \\
&= \big(\overline{X}(\overline{g}(\overline{Y}, \overline{Z})) - \overline{g}(\tan\overline{\nabla}_{\overline{X}}\overline{Y}, \overline{Z}) - \overline{g}(\overline{Y}, \tan\overline{\nabla}_{\overline{X}}\overline{Z})\big)\big|_p \\
&= \big(\overline{X}(\overline{g}(\overline{Y}, \overline{Z})) - \overline{g}(\overline{\nabla}_{\overline{X}}\overline{Y}, \overline{Z}) - \overline{g}(\overline{Y}, \overline{\nabla}_{\overline{X}}\overline{Z})\big)\big|_p
\end{aligned}
$$

- hence, $\overline{\nabla}\overline{g} = 0$ implies that that $\tan\overline{\nabla} g = 0$, showing the **metric property**

(continued on next page)

(continuation of proof)

- for **torsion-freeness**, calculate for all $X, Y \in \mathfrak{X}(M)$ with arbitrary extensions $\overline{X}, \overline{Y} \in \mathfrak{X}(\overline{M})$ and all $p \in M$ with the help of Lemmas A & B

$$\begin{aligned}
&\left(\tan\overline{\nabla}_X Y - \tan\overline{\nabla}_Y X - [X, Y]\right)\Big|_p \\
&= \left(\tan\overline{\nabla}_{\overline{X}}\overline{Y} - \tan\overline{\nabla}_{\overline{Y}}\overline{X} - [\overline{X}, \overline{Y}]\right)\Big|_p \\
&= \tan\left(\overline{\nabla}_{\overline{X}}\overline{Y} - \overline{\nabla}_{\overline{Y}}\overline{X} - [\overline{X}, \overline{Y}]\right)\Big|_p = 0
\end{aligned}$$

as required □

Remark

Proposition A allows us to calculate **covariant derivatives in pseudo-Riemannian submanifolds** with respect to the Levi-Civita connection using **only tangential projections** and the **Levi-Civita connection of the ambient manifold**. The latter is usually **easier to handle**.

By a result from John Nash in *The Imbedding Problem for Riemannian Manifolds*, Annals of Mathematics, Second Series, Vol. **63**, No. **1**, calculating Levi-Civita connections as in Proposition A is actually the **most general case for Riemannian manifolds**:

Theorem

Let (M, g) be a **Riemannian manifold** of dimension n. Then there exists an **isometric embedding** of (M, g) into any open subset $U \subset \mathbb{R}^m$ for $m = \frac{3}{2}n^3 + 7n^2 + \frac{11}{2}n$ equipped with the standard Riemannian metric $\langle \cdot, \cdot \rangle$.

Note: In reality, constructing such isometric embeddings **explicitly** for given $U \subset \mathbb{R}^m$ is **far from trivial**. A picture one might have in mind for $M = (0, 1)$ and $m = 2$ is "rolling up" the open interval tight enough to fit into $U \subset \mathbb{R}^2$.

Solution 2:

Proposition B

Let (M, g) be a pseudo-Riemannian submanifold of $(\overline{M}, \overline{g})$ and let $\overline{\nabla}$ denote the Levi-Civita connection of $(\overline{M}, \overline{g})$. Then

$$\overline{\nabla}^{\mathrm{nor}} := \mathrm{nor}\overline{\nabla} : \mathfrak{X}(M) \times \mathfrak{X}(M)^{\perp} \to \mathfrak{X}(M)^{\perp}, \quad (X, \xi) \mapsto \mathrm{nor}\overline{\nabla}_X \xi,$$

for all $X \in \mathfrak{X}(M)$ and all $\xi \in \mathfrak{X}(M)^{\perp}$ **is a connection in** $TM^{\perp} \to M$, called the **normal connection**.

Proof: Follows from the **fibrewise linearity** of nor and Lemma B. □

Next, we will study the **difference** between the Levi-Civita connections of a pseudo-Riemannian submanifold and its ambient manifold restricted to the tangent part of the submanifold. We will need the following definition:

Definition

Let (M, g) be a pseudo-Riemannian submanifold of $(\overline{M}, \overline{g})$ with Levi-Civita connection $\overline{\nabla}$ in $T\overline{M} \to \overline{M}$. The **second fundamental form** of M is defined as

$$\mathrm{II} : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)^{\perp}, \quad \mathrm{II}(X, Y) := \mathrm{nor}\overline{\nabla}_X Y$$

for all $X, Y \in \mathfrak{X}(M)$.

The second fundamental form has the following properties:

Lemma

The second fundamental form is a **symmetric** $TM^{\perp}$**-valued** $(0, 2)$-**tensor field**, that is a section of $TM^{\perp} \otimes \mathrm{Sym}^2(T^*M) \to M$.

Proof: (next page)

(continuation of proof)

- II is $C^\infty(M)$-**linear** in its **first argument**
- hence: it suffices to show that II is **symmetric** in order to also obtain the $C^\infty(M)$-**linearity** in the **second argument**
- $\rightsquigarrow$ for the **symmetry** we check that for all $X, Y \in \mathfrak{X}(M)$ using the **fibrewise linearity of** nor, the **torsion-freeness of** $\overline{\nabla}$, and Lemma A

$$II(X,Y) - II(Y,X) = \mathrm{nor}\left(\overline{\nabla}_X Y - \overline{\nabla}_Y X\right) = \mathrm{nor}[X,Y] = 0$$

□

Corollary

The Levi-Civita connection ∇ and second fundamental form II of a pseudo-Riemannian submanifold (M, g) of $(\overline{M}, \overline{g})$ fulfil the **Gauß equation**

$$\overline{\nabla}_X Y = \nabla_X Y + II(X,Y)$$

for all $X, Y \in \mathfrak{X}(M)$.

Warning: There are **many** "Gauß equations", we will encounter one more formula with this name.

Example

The second fundamental form of the **graph of a smooth function** $f : \mathbb{R} \to \mathbb{R}$ fulfils for canonical coordinates (x, y) of the ambient space $(\mathbb{R}^2, \langle\cdot,\cdot\rangle)$

$$\mathrm{II}\left(\frac{\partial}{\partial x} + \frac{\partial f}{\partial x}\frac{\partial}{\partial y}, \frac{\partial}{\partial x} + \frac{\partial f}{\partial x}\frac{\partial}{\partial y}\right) = \frac{\frac{\partial^2 f}{\partial x^2}}{\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2}}\xi,$$

where $\xi = \frac{1}{\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2}}\left(-\frac{\partial f}{\partial x}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)$.

The covariant derivatives of **normal fields** along pseudo-Riemannian manifolds **also split** in a certain manner. For this we need the following definition: (see next page)

Definition

Let (M, g) be a pseudo-Riemannian submanifold of $(\overline{M}, \overline{g})$. For all $\xi \in \mathfrak{X}(M)^\perp$, the g-**symmetric endomorphism field** $S^\xi \in \mathcal{T}^{1,1}(M)$ defined by the **Weingarten equation**

$$\overline{g}(II(X, Y), \xi) = g(S^\xi X, Y)$$

for all $X, Y \in \mathfrak{X}(M)$ is called **Weingarten map** (alternatively **shape operator**).

Note: S^ξ is well-defined for each $\xi \in \mathfrak{X}(M)^\perp$ by the **fibrewise nondegeneracy** of g.

The Weingarten map fulfils the following **identity**:

Proposition C

Let (M, g) be a pseudo-Riemannian submanifold of $(\overline{M}, \overline{g})$. The Weingarten map fulfils the **Weingarten equation**

$$\overline{\nabla}_X \xi = -S^{\xi} X + \overline{\nabla}^{\mathrm{nor}}_X \xi$$

for all $X \in \mathfrak{X}(M)$ and all $\xi \in \mathfrak{X}(M)^{\perp}$.

Proof: Follows by **writing out** $\overline{g}(\overline{\nabla}_X \xi, \overline{Y})$ for $\overline{Y} \in \mathfrak{X}(\overline{M})$ with $\overline{Y}|_M \in \mathfrak{X}(M)$ arbitrary and using Proposition B and the **metric property** of $\overline{\nabla}$. □

Question: How are the **Riemann curvature tensors** of a **pseudo-Riemannian submanifold** and its **ambient manifold** related?
Answer:

Proposition

Let (M, g) be a pseudo-Riemannian submanifold of $(\overline{M}, \overline{g})$.
The **Riemann curvature tensors** R of (M, g) and $\overline{R}$ of $(\overline{M}, \overline{g})$ are **related** by the **Gauß equation for Riemann curvature tensors**

$$\begin{aligned} &\overline{g}(\overline{R}(X, Y)Z, W) \\ &= g(R(X, Y)Z, W) \\ &+ \overline{g}(\mathrm{II}(X, Z), \mathrm{II}(Y, W)) - \overline{g}(\mathrm{II}(Y, Z), \mathrm{II}(X, W)) \end{aligned}$$

for all $X, Y, Z, W \in \mathfrak{X}(M)$.

Proof:

- since above equation is a **tensor equation**, we might **without loss of generality assume** $[X, Y] = 0$

(continued on next page)

(continuation of proof)

- observe that for all $X, Y, Z, W \in \mathfrak{X}(M)$, Proposition C implies

$$
\begin{aligned}
&\overline{g}(\overline{\nabla}_X \overline{\nabla}_Y Z, W) \\
&= \overline{g}(\tan \overline{\nabla}_X \overline{\nabla}_Y Z, W) \\
&= \overline{g}(\nabla_X(\tan \overline{\nabla}_Y Z), W) + \overline{g}(\tan \overline{\nabla}_X(\mathrm{nor}\, \overline{\nabla}_Y Z), W) \\
&= \overline{g}(\nabla_X \nabla_Y Z, W) + \overline{g}(\tan \overline{\nabla}_X(\mathrm{II}(Y, Z)), W) \\
&= \overline{g}(\nabla_X \nabla_Y Z, W) - \overline{g}(S^{\mathrm{II}(Y,Z)} X, W) \\
&= \overline{g}(\nabla_X \nabla_Y Z, W) - \overline{g}(\mathrm{II}(X, W), \mathrm{II}(Y, Z)).
\end{aligned}
$$

- repeating the above calculation with X and Y **interchanged** and using $\overline{R}(X, Y) = \overline{\nabla}_X \overline{\nabla}_Y - \overline{\nabla}_Y \overline{\nabla}_X$ for $[X, Y] = 0$ we obtain the claimed formula □

As a consequence we obtain a formula for the relation of the **sectional curvatures** of pseudo-Riemannian submanifolds and their ambient manifolds.

Corollary

Let (M, g) be a pseudo-Riemannian submanifold of $(\overline{M}, \overline{g})$. For every nondegenerate plane spanned by $v, w \in T_pM$ in $T_pM \subset \ T_p\overline{M}$, the **sectional curvatures** K of (M, g) and $\overline{K}$ of $(\overline{M}, \overline{g})$ **are related by**

$$\overline{K}(v, w) = K(v, w) - \frac{\overline{g}(\mathrm{II}(v, v), \mathrm{II}(w, w)) - \overline{g}(\mathrm{II}(v, w), \mathrm{II}(v, w))}{g(v, v)g(w, w) - g(v, w)^2}.$$

Note: The above corollary is particularly nice if the ambient manifold has **constant sectional curvature**.

Example

The **unit sphere** $S^n \subset \mathbb{R}^{n+1}$ equipped with the restriction of the standard Riemannian metric has **constant sectional curvature** 1.

We have seen how to relate the **Levi-Civita connections** and the **curvature tensors** of pseudo-Riemannian manifolds and their pseudo-Riemannian submanifolds. Next, we will study **geodesics of pseudo-Riemannian submanifolds**.

Proposition D

Let (M, g) be a pseudo-Riemannian submanifold of $(\overline{M}, \overline{g})$ with **respective Levi-Civita connections** ∇ **and** $\overline{\nabla}$. A smooth curve $\gamma : I \to M$ is a **geodesic with respect to** ∇ if and only if $\overline{\nabla}_{\gamma'}\gamma'$ **is normal at every point**, i.e. $\overline{\nabla}_{\gamma'}\gamma' \in \Gamma_\gamma(TM^\perp)$.

Proof:

- **Gauß equation** for the Levi-Civita connections $\rightsquigarrow$

$$\overline{\nabla}_{\gamma'}\gamma' = \nabla_{\gamma'}\gamma' + \mathrm{II}(\gamma', \gamma')$$

- since $\mathrm{II}(\gamma', \gamma')$ is precisely the **normal part** of $\overline{\nabla}_{\gamma'}\gamma'$, the claim follows $\square$

Remark

Proposition D usually **makes life easier** in case the ambient manifold is the **flat Euclidean space**.

Example

Geodesics in the unit sphere $S^n \subset \mathbb{R}^{n+1}$ are precisely curves of constant velocity contained in the **great circles**.

Note: In the above example, geodesics in the Riemannian submanifold S^n are **not** geodesics in the ambient space $(\mathbb{R}^{n+1}, \langle \cdot, \cdot \rangle)$.
Question: What conditions must a pseudo-Riemannian submanifold fulfil so that **its geodesics are also geodesics in the ambient manifold**?
Answer: (see next page)

Definition

A pseudo-Riemannian submanifold (M, g) of a pseudo-Riemannian manifold $(\overline{M}, \overline{g})$ is called **totally geodesic** if all geodesics $\gamma : I \to \overline{M}$ of $(\overline{M}, \overline{g})$ **starting in** M with **initial velocity tangent to** M stay in M for all time, i.e. $\gamma(I) \subset M$.

From the Gauß equation for connections we obtain the following result to **check** whether a pseudo-Riemannian submanifold is totally geodesic or not.

Lemma C

A pseudo-Riemannian submanifold (M, g) of a pseudo-Riemannian manifold $(\overline{M}, \overline{g})$ is totally geodesic **if and only** if its second fundamental form **vanishes identically**.

Proof:

- $\mathrm{II} \equiv 0 \rightsquigarrow$ Gauß equation implies that $\overline{\nabla}_{\gamma'} \gamma' = \nabla_{\gamma'} \gamma'$ **for all smooth curves** $\gamma : I \to M$

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(continuation of proof)

- hence: γ is a geodesic in (M, g) **if and only if** it is a geodesic in $(\overline{M}, \overline{g})$
- for the other direction suppose that $\mathrm{II} \not\equiv 0$
- **fix** $p \in M$ and $v \in T_pM \subset T_p\overline{M}$, such that $\mathrm{II}(v, v) \neq 0$
- for $\varepsilon > 0$ small enough let $\gamma : (-\varepsilon, \varepsilon) \to \overline{M}$ be a geodesic in (M, g) with $\gamma'(0) = v$
- Gauß equation $\rightsquigarrow \gamma$ is **not a geodesic** in $(\overline{M}, \overline{g})$
- hence, (M, g) is **not totally geodesic** $\square$

In order to check if a pseudo-Riemannian submanifold is total geodesic, we might use the following **equivalent conditions**:

Proposition E

A ps.-R. submanifold $(M, g) \subset (\overline{M}, \overline{g})$ is totally geodesic **if and only if** one of the following **equivalent statements** hold:

i. Every geodesic in (M, g) is a geodesic in $(\overline{M}, \overline{g})$.

ii. For every geodesic $\gamma : I \to \overline{M}$ with $0 \in I$, I open, and initial conditions $\gamma(0) = p$, $\gamma'(0) = v \in T_pM \subset T_p\overline{M}$ there exists $\varepsilon > 0$, such that $\gamma((-\varepsilon, \varepsilon)) \subset M$.

iii. For every smooth curve $\gamma : [a, b] \to M \subset \overline{M}$ the parallel transport in (M, g),

$$P_a^b(\gamma) : T_{\gamma(a)}M \to T_{\gamma(b)}M$$

coincides with the parallel transport in $(\overline{M}, \overline{g})$,

$$\overline{P}_a^b : T_{\gamma(a)}\overline{M} \to T_{\gamma(b)}\overline{M}$$

restricted to $T_{\gamma(a)}M \subset T_{\gamma(a)}\overline{M}$.

Proof:

- (i) is **by definition** of totally geodesic pseudo-Riemannian submanifolds equivalent to (M, g) being totally geodesic
- (iii) is equivalent to (M, g) being totally geodesic by Lemma C
- "(i) $\Rightarrow$ (ii)": let $\gamma : I \to \overline{M}$ be a **geodesic in** $(\overline{M}, \overline{g})$ with $0 \in I$, I open, and $\gamma'(0) = v \in T_pM$, and let $\widetilde{\gamma} : \widetilde{I} \to M$ be a **geodesic in** (M, g) with $0 \in \widetilde{I}$, $\widetilde{I}$ open, and $\widetilde{\gamma}'(0) = v$
- by assumption, $\widetilde{\gamma}$ **is also a geodesic in** $(\overline{M}, \overline{g})$ and by the **uniqueness property** of maximal geodesics **coincides with** γ on $I \cap \widetilde{I}$
- choosing $\varepsilon > 0$ **small enough** so that $(-\varepsilon, \varepsilon) \subset I \cap \widetilde{I}$ proves the claim
- "(ii) $\Rightarrow$ (i)": let $p \in M$ and $v \in T_pM$ be arbitrary and $\gamma : (-\varepsilon, \varepsilon) \to \overline{M}$ for any $\varepsilon > 0$ **small enough** be a **geodesic in** $(\overline{M}, \overline{g})$ with $\gamma(0) = p$, $\gamma'(0) = v$, such that $\gamma((-\varepsilon, \varepsilon)) \subset M$

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(continuation of proof)

- Gauß equation $\rightsquigarrow$

$$0 = \overline{\nabla}_{\gamma'}\gamma'\big|_{t=0} = \nabla_{\gamma'}\gamma'\big|_{t=0} + \mathrm{II}_p(v, v)$$

- hence, the fact that we have a **fibrewise direct sum** of the splitting $T\overline{M}|_M = TM \oplus TM^\perp$ and p and v were **arbitrary** implies $\mathrm{II} \equiv 0$ as required □

Question: How can one **construct** totally geodesic submanifolds?
Answer: One neat possibility is the following:

Proposition

Let $(\overline{M}, \overline{g})$ be a pseudo-Riemannian manifold and let $F \in \mathrm{Isom}(\overline{M}, \overline{g})$ be an **isometry of** $(\overline{M}, \overline{g})$. Suppose that a **connected component** M **of** $\mathrm{Fix}(F) := \{p \in \overline{M} \mid F(p) = p\}$ is a **pseudo-Riemannian submanifold** of $(\overline{M}, \overline{g})$. Then M is **totally geodesic**.

Proof:

- F restricted to M is the **identity**, i.e. $F|_M = \mathrm{id}_M$
- hence, dF restricted to the subbundle $TM \subset T\overline{M}|_M$ is **also the identity**, meaning that $dF(v) = v$ for all $v \in T_pM \subset T_p\overline{M}$ and all $p \in M$

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(continuation of proof)

- to show that M is **totally geodesic**, it **suffices** to show by Proposition E (ii) that **isometries map geodesics** of $(\overline{M}, \overline{g})$ **to geodesics** of $(\overline{M}, \overline{g})$
- this means that for $\varepsilon > 0$ **small enough**, any geodesic of $(\overline{M}, \overline{g})$, $\gamma : (-\varepsilon, \varepsilon) \to \overline{M}$ with $\gamma(0) = p \in M$, $\gamma'(0) = v \in T_pM$, will be **contained** in M by construction
- $\rightsquigarrow$ follows from the **naturality** of the Levi-Civita connection, that is $F_* \overline{\nabla}_{\overline{X}} \overline{Y} = \overline{\nabla}_{F_* \overline{X}} F_* \overline{Y}$ for all $\overline{X}, \overline{Y} \in \mathfrak{X}(\overline{M})$ □

In the case that considered totally geodesic submanifolds are **geodesically complete** and connected, we have the following convenient way to check if they are **isometric**:

Proposition

Let M and N be **connected totally geodesic geodesically complete** pseudo-Riemannian submanifolds of $(\overline{M}, \overline{g})$. If there exists $p \in M \cap N$, such that $T_pM = T_pN$ as linear subspaces of $T_p\overline{M}$, we already have $M = N$.

Proof:

- we will show $M \subset N$, the other direction follows by **symmetry** of the arguments
- let $\gamma : [a, b] \to M$ be a **geodesic in** M from $p = \gamma(a)$ to $q := \gamma(b)$ and note that M being totally geodesic implies γ is **also a geodesic in** $\overline{M}$

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(continuation of proof)

- by assumption of **geodesic completeness of N**, there exists a **unique geodesic**

$$\widetilde{\gamma} : \mathbb{R} \to N$$

  with $\widetilde{\gamma}'(a) = \gamma'(a)$
- since $N \subset \overline{M}$ is totally geodesic, $\widetilde{\gamma}$ is **also a geodesic in $\overline{M}$**
- hence by **uniqueness of maximal geodesics** γ and $\widetilde{\gamma}|_{[a,b]}$ **coincide**, showing in particular $q \in N$
- By Proposition E (iii) and the **linear isometry property** of $P_a^b(\gamma) : T_pM \to T_qM$ it follows that $T_qM = T_qN$
- by the **connectedness of M and N** and the fact that we can connect arbitrary points in connected geodesically complete manifolds with **piecewise geodesics**, we conclude that this argument holds **for all $q \in M$**, showing that $q \in N$ $\square$

Corollary

The **connected totally geodesic geodesically complete** Riemannian submanifolds of $\mathbb{R}^n$ with standard Riemannian metric are the **affine** $m \leq n$**-spaces**, that is smooth submanifolds of the form

$$M = p + V, \quad V \subset \mathbb{R}^n \text{ } m\text{-dimensional linear subspace.}$$

Lastly, we focus our studies on **pseudo-Riemannian hypersurfaces**, that is pseudo-Riemannian submanifolds of codim. 1.

Definition

Let $(\overline{M}, \overline{g})$ be a pseudo-Riemannian manifold with **orientable pseudo-Riemannian hypersurface** (M, g). An orthogonal vector field $\xi \in \mathfrak{X}(M)^{\perp}$ is called **unit normal** if $\overline{g}(\xi, \xi) \equiv 1$ or $\overline{g}(\xi, \xi) \equiv -1$.

Note: Orientability of hypersurfaces is usually defined by requiring that there exists a **nowhere vanishing transversal vector field** along said hypersurface. Alternatively, one can study the existence of a globally defined **volume form**.

If a pseudo-Riemannian hypersurface **admits** a unit normal, it admits **precisely** 2 unit normals related by a sign flip.

Proposition

Let (M, g) be an oriented pseudo-Riemannian hypersurface of $(\overline{M}, \overline{g})$ and let $\xi \in \mathfrak{X}(M)^{\perp}$ be a **unit normal** with $\overline{g}(\xi, \xi) \equiv \varepsilon \in \{-1, 1\}$. Then the **second fundamental form of** M is of the form

$$\mathrm{II} = \xi \otimes \widetilde{g},$$

where $\widetilde{g} \in \Gamma(\mathrm{Sym}^2(T^*M))$ is given by

$$\widetilde{g}(X, Y) = \varepsilon g(S^{\xi} X, Y)$$

for all $X, Y \in \mathfrak{X}(M)$ with S^{ξ} the **Weingarten map**.

Proposition (continuation)

The **Gauß equation for the curvature** and the related **sectional curvature equation** are given by

$$\overline{g}(\overline{R}(X,Y)Z,W) = g(R(X,Y)Z,W) + \varepsilon(g(S^\xi X,Z)g(S^\xi Y,W) - g(S^\xi Y,Z)g(S^\xi X,W)),$$

$$\overline{K}(v,w) = K(v,w) - \varepsilon\frac{g(S^\xi v,v)g(S^\xi w,w) - g(S^\xi v,w)^2}{g(v,v)g(w,w) - g(v,w)^2},$$

for all $X,Y,Z,W \in \mathfrak{X}(M)$ and all $v,w \in T_pM$ spanning a **nondegenerate plane**.

Proof:

- ξ being by assumption **nowhere vanishing** implies that we can write $\mathrm{II} = \xi \otimes \widetilde{g}$
- $\overline{g}(\mathrm{II}(X,Y),\xi) = \varepsilon\widetilde{g}(X,Y)$ for all $X,Y \in \mathfrak{X}(M)$ means that $\widetilde{g}$ is **uniquely determined**

(continued on next page)

(continuation of proof)

- the **Weingarten equation** is given by

$$\overline{g}(\mathrm{II}(X,Y),\xi) = g(S^{\xi}X,Y)$$

  for all $X, Y \in \mathfrak{X}(M)$ and, hence, $\widetilde{g}(X,Y) = \varepsilon g(S^{\xi}X,Y)$ **as claimed**

- for the **curvature equations** in this proposition observe that

$$\begin{aligned}\overline{g}(\mathrm{II}(X,Z),\mathrm{II}(Y,W)) &= \varepsilon\widetilde{g}(X,Z)\widetilde{g}(Y,W)\\ &= \varepsilon g(S^{\xi}X,Z)g(S^{\xi}Y,W),\end{aligned}$$

  for all $X, Y, Z, W \in \mathfrak{X}(M)$, which follows from $\varepsilon = \varepsilon^{-1}$ and our **previous results**

- the rest of this proof is just **writing out the formulas**, that is the Gauß equation for R and $\overline{R}$ and the sectional curvature relation **[**Exercise!**]** □

Example

Let (M, g) be an **orientable Riemannian hypersurface** in $(\mathbb{R}^{n+1}, \langle\cdot,\cdot\rangle)$. For $p \in M$ and $U \subset \mathbb{R}^{n+1}$ a small enough open neighbourhood of p, choose $f \in C^\infty(U)$ of **maximal rank**, such that

$$M \cap U = \{f = 0\}.$$

After a possible overall sign flip of f, we can assume w.l.o.g. that the **unit normal** of M is given locally on $M \cap U$ by

$$\xi = \frac{\mathrm{grad}_{\langle\cdot,\cdot\rangle}(f)}{\sqrt{\langle \mathrm{grad}_{\langle\cdot,\cdot\rangle}(f), \mathrm{grad}_{\langle\cdot,\cdot\rangle}(f)\rangle}} = \frac{\mathrm{grad}_{\langle\cdot,\cdot\rangle}(f)}{\|\mathrm{grad}_{\langle\cdot,\cdot\rangle}(f)\|}.$$

The **second fundamental form** of $M \cap U$ is then given by

$$\mathrm{II}(X, Y) = -\frac{1}{\|\mathrm{grad}_{\langle\cdot,\cdot\rangle}(f)\|} \overline{\nabla}^2 f(X, Y)$$

for all $X, Y \in \mathfrak{X}(M \cap U)$, where $\overline{\nabla}^2 f$ denotes the **Hessian of** f w.r.t. $\overline{\nabla}$.

END OF LECTURE 19