Differential geometry
Lecture 18: Curvature

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14. July 2020
1. Riemann curvature tensor

2. Sectional curvature

3. Ricci curvature

4. Scalar curvature
Recap of lecture 17:

- defined **geodesics** in pseudo-Riemannian manifolds as curves with parallel velocity
- viewed geodesics as **projections of integral curves** of a vector field $G \in \mathfrak{X}(TM)$ with local flow called **geodesic flow**
- obtained **uniqueness** and **existence** properties of geodesics
- constructed the **exponential map** $\exp : V \rightarrow M$, $V$ neighbourhood of the zero-section in $TM \rightarrow M$
- showed that geodesics with compact domain are precisely the **critical points** of the **energy functional**
- used the exponential map to construct **Riemannian normal coordinates**, studied local forms of the metric and the Christoffel symbols in such coordinates
- discussed the **Hopf-Rinow Theorem**
- **erratum**: codomain of $(x, v)$ as **local integral curve** of $G$ is $d\varphi(TU)$, not $TM$
Intuitively, a meaningful definition of the term “curvature” for a smooth surface in $\mathbb{R}^3$, written locally as a graph of a smooth function $f : U \subset \mathbb{R}^2 \to \mathbb{R}$, should involve the second partial derivatives of $f$ at each point. How can we find a coordinate-free definition of curvature not just for surfaces in $\mathbb{R}^3$, which are automatically Riemannian manifolds by restricting $\langle \cdot, \cdot \rangle$, but for all pseudo-Riemannian manifolds?

**Definition**

Let $(M, g)$ be a pseudo-Riemannian manifold with Levi-Civita connection $\nabla$. The **Riemann curvature tensor** of $(M, g)$ is defined as

$$R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M),$$

$$R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z$$

for all $X, Y, Z \in \mathfrak{X}(M)$. In the above formula, we understand $\nabla_X \nabla_Y Z$ as $\nabla_X (\nabla_Y Z)$, analogously for $X$ and $Y$ interchanged.
The first thing we need to check is if $R$ is, as implied in its definition, actually a tensor field:

**Lemma**

The Riemann curvature tensor is, in fact, a tensor field, i.e. $R \in \mathcal{T}^{1,3}(M)$.

**Proof:** Direct calculation.

**Note:** If we would replace $\nabla$ with $\mathcal{L}$ in the definition of $R$, it would identically vanish by the Jacobi identity. Also observe that the Riemann curvature tensor vanishes identically if $\dim(M) = 1$.

**Question:** Why should we study the Riemann curvature tensor $R$ in the first place? What is the geometric picture one should have in mind for $R$?

**(partial) Answer:** (see next page)
Lemma A

Let \((M, g)\) be a pseudo-Riemannian manifold with Levi-Civita connection \(\nabla\). For any \(X \in \mathfrak{X}(M)\), denote for every \(p \in M\) by \(P_{0}^{t}(X) : T_{p}M \to T_{\gamma(t)}M\) the parallel transport map with respect to \(\nabla\) along the integral curve \(\gamma : (-\varepsilon, \varepsilon) \to M\) of \(X\) with \(\gamma(0) = p\) for \(\varepsilon > 0\) small enough, that is for \(p \in M\) fixed we have \(P_{0}^{t}(X) = P_{0}^{t}(\gamma)\). Let \((x^{1}, \ldots, x^{n})\) be local coordinates on \(U \subset M\). Then

\[
R\left(\frac{\partial}{\partial x^{i}}\bigg|_{p}, \frac{\partial}{\partial x^{j}}\bigg|_{p}\right) \frac{\partial}{\partial x^{k}}\bigg|_{p} = \\
\frac{\partial}{\partial s}\bigg|_{s=0} \frac{\partial}{\partial t}\bigg|_{t=0} P_{0}^{s}\left(\frac{\partial}{\partial x^{i}}\right)^{-1} P_{0}^{t}\left(\frac{\partial}{\partial x^{j}}\right)^{-1} P_{0}^{s}\left(\frac{\partial}{\partial x^{i}}\right) P_{0}^{t}\left(\frac{\partial}{\partial x^{j}}\right) \frac{\partial}{\partial x^{k}}\bigg|_{p}
\]

for all \(1 \leq i, j, k \leq n\) and all \(p \in U\). The Riemann curvature tensor is the unique \((1, 3)\)-tensor field fulfilling the above equation in all local coordinates.

Proof: (next page)
(continuation of proof)

- in coordinate representations,
  \[ P^t_0 \left( \frac{\partial}{\partial x^j} \right) : T_p M \to T_{\gamma(t)} M, \ t \in (-\varepsilon, \varepsilon), \text{ and the other parallel translations are smooth maps of the form} \]
  \[ P^t_0 \left( \frac{\partial}{\partial x^j} \right) : (-\varepsilon, \varepsilon) \to GL(n), \]

  where \( GL(n) \) being the codomain follows the fact that parallel translations are isometries, hence isomorphisms, for each fixed \( t \)

- the above map should be understood as mapping prefactors of vectors in \( T_p M \) written in the coordinate basis to prefactors of vectors in \( T_{\gamma(t)} M \), again written in the coordinate basis

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(continuation of proof)

- hence, the **partial derivatives of products** of \( P^t_0 \left( \frac{\partial}{\partial x^i} \right) \) behave according to the **product rule of matrix valued curves**, i.e. for all \( A, B : (-\varepsilon, \varepsilon) \to \text{GL}(n) \) smooth with \( A(0) = B(0) = 1 \) and all \( v \in \mathbb{R}^n \) we have

\[
\left. \frac{\partial}{\partial s} \right|_{s=0} A(s)^{-1} B(t)^{-1} A(s) B(t) v \\
= \left( \left. \frac{\partial}{\partial s} \right|_{s=0} A(s)^{-1} \right) \left( \left. \frac{\partial}{\partial t} \right|_{t=0} B(t)^{-1} \right) v \\
+ \left( \left. \frac{\partial}{\partial t} \right|_{t=0} B(t)^{-1} \right) \left( \left. \frac{\partial}{\partial s} \right|_{s=0} A(s) \right) v \\
= \left( \left. \frac{\partial}{\partial s} \right|_{s=0} A(s) \right) \left( \left. \frac{\partial}{\partial t} \right|_{t=0} B(t) \right) v \\
- \left( \left. \frac{\partial}{\partial t} \right|_{t=0} B(t) \right) \left( \left. \frac{\partial}{\partial s} \right|_{s=0} A(s) \right) v
\]

- note that \( \left. \frac{\partial}{\partial s} \right|_{s=0} A(s) \in \text{End}(\mathbb{R}^n) \), meaning that the derivative is in general **not invertible**

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hence, using our formula relating $\nabla$ with parallel transport maps, obtain

$$
\frac{\partial}{\partial s}\bigg|_0 \frac{\partial}{\partial \tau}\bigg|_0 P_0^s \left( \frac{\partial}{\partial x^i} \right)^{-1} P_0^t \left( \frac{\partial}{\partial x^j} \right)^{-1} P_0^s \left( \frac{\partial}{\partial x^i} \right) P_0^t \left( \frac{\partial}{\partial x^j} \right) \frac{\partial}{\partial x^k} \bigg|_p
$$

$$=
\left( \nabla \frac{\partial}{\partial x^i} \nabla \frac{\partial}{\partial x^j} \frac{\partial}{\partial x^k} - \nabla \frac{\partial}{\partial x^i} \nabla \frac{\partial}{\partial x^j} \frac{\partial}{\partial x^k} \right) \bigg|_p
$$

proving the first statement of this lemma

in order to show that $R$ is indeed the unique tensor field fulfilling the above, we only need to check that for any local functions $X^1, Y^1, Z^1, \ldots, X^n, Y^n, Z^n \in C^\infty(U)$ and all $p \in U$, $\sum_{i,j,k} X^i(p) Y^j(p) Z^k(p) R \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k} \right) \bigg|_p$

and $(R(X,Y)Z)|_p$ via its initial definition coincide

[Exercise!]
We have the following **local formula** for the Riemann curvature tensor:

**Lemma**

In **local coordinates** \((x^1, \ldots, x^n)\) the Riemann curvature tensor of a pseudo-Riemannian manifold \((M, g)\) has components

\[
R^\ell_{ijk} := dx^\ell \left( R \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) \frac{\partial}{\partial x^k} \right),
\]

so that **locally** \(R = \sum_{i,j,k,\ell} R^\ell_{ijk} \frac{\partial}{\partial x^\ell} \otimes dx^i \otimes dx^j \otimes dx^k\). The local functions \(R^\ell_{ijk}\) are given by

\[
R^\ell_{ijk} = \frac{\partial \Gamma^\ell_{jk}}{\partial x^i} - \frac{\partial \Gamma^\ell_{ik}}{\partial x^j} + \sum_{m=1}^{n} \left( \Gamma^m_{im} \Gamma^\ell_{jk} - \Gamma^m_{jm} \Gamma^\ell_{ik} \right)
\]

for all \(1 \leq i, j, k, \ell \leq n\).

**Proof:** Direct calculation.
The Riemann curvature tensor $R$ of a pseudo-Riemannian manifold $(M, g)$ fulfills the following identities:

### Lemma

1. $R(X, Y) = -R(Y, X)$,
2. $g(R(X, Y)Z, W) = -g(Z, R(X, Y)W)$,
3. $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$ (first or algebraic Bianchi identity),
4. $g(R(X, Y)Z, W) = g(R(Z, W)X, Y),
5. $(\nabla_X R)(Y, Z) + (\nabla_Y R)(Z, X) + (\nabla_Z R)(X, Y) = 0$ (second or differential Bianchi identity)

for all $X, Y, Z, W \in \mathfrak{X}(M)$.

**Proof:** (see notes on the right hand side)
As one might expect from the tensoriality, the Riemann curvature tensor behaves well under isometries.

**Lemma B**

Let $F : (M, g) \to (N, h)$ be an isometry and let $R^M$ and $R^N$ denote the Riemann curvature tensors of $(M, g)$ and $(N, h)$, respectively. Then

$$F_*(R^M(X, Y)Z) = R^N(F_*X, F_*Y)F_*Z$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

**Proof:**

- suffices to show that $F_* \nabla^M_X Y = \nabla^N_{F_*X}(F_*Y)$ for all $X, Y \in \mathfrak{X}(M)$, where $\nabla^M$ and $\nabla^N$ denote the Levi-Civita connections of $(M, g)$ and $(N, h)$, respectively

- use Koszul formula for $\nabla^M$ and $\nabla^N$, bijectivity of $F_* : \mathfrak{X}(M) \to \mathfrak{X}(N)$, and $F_*[X, Y] = [F_*X, F_*Y]$
Definition
A pseudo-Riemannian manifold with vanishing Riemann curvature tensor is called flat.

Examples
The following pseudo-Riemannian manifolds are flat:
- \((\mathbb{R}^n, \langle \cdot, \cdot \rangle_\nu), 0 \leq \nu \leq n\)
- the cylinder \(\mathbb{R} \times S^1\) and the 2-torus \(T^2 = S^1 \times S^1\) equipped with the respective product metric
- more generally, \((M \times N, g + h)\) for all \((M, g)\) and \((N, h)\) flat
Suppose that we are given a flat Riemannian manifold \((M, g)\).

**Question:** Is \((M, g)\) automatically of a simple form, at least locally (up to isometry)?

**Answer:** Yes! **Locally**, we have the following result:

### Theorem

An \(n\)-dimensional Riemannian manifold \((M, g)\) is flat if and only if it is **locally isometric** to \((\mathbb{R}^n, \langle \cdot, \cdot \rangle)\), meaning that for all \(p \in M\) there exists an open neighbourhood \(U \subset M\) of \(p\) and an isometry \(F : (U, g) \to (F(U), \langle \cdot, \cdot \rangle)\), \(F(U) \subset \mathbb{R}^n\) open.

**Proof:**

- Lemma B \(\Rightarrow\) **local isometry** to \((\mathbb{R}^n, \langle \cdot, \cdot \rangle)\) implies **flatness**, i.e. \(R \equiv 0\)

- the other direction of this proof requires a lot more work, for details see Theorem 7.3 in J.M. Lee’s *Riemannian Manifolds – An Introduction to Curvature*, Springer GTM 176 (1997) (with slightly different conventions)

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(continuation of proof)

- The idea is to construct a **commuting orthonormal local frame** of $TM \to M$ near every given point.

- The key ingredient is that **parallel transport of vectors** at, say, $p \in M$, to a close enough point $q \in M$ does not depend on the chosen curve starting at $p$ and ending at $q$ if it is required to be contained in a small enough open neighbourhood of both $p$ and $q$.

- Follows from a **similar argument** as in Lemma A.
In Lemma A we have described how to interpret the Riemann curvature tensor geometrically as infinitesimal change of parallel transport of tangent vectors around infinitesimal parallelograms.

**Question:** Is there another motivation for the definition of the Riemann curvature tensor?

**Answer:** Yes, via second covariant derivatives!

**Definition**

Let $(M, g)$ be a pseudo-Riemannian manifold with Levi-Civita connection $\nabla$. Then for all $X, Y, Z \in \mathfrak{X}(M)$,

$$\nabla^2_{X, Y} Z := (\nabla_X (\nabla Z))(Y) = \nabla_X \nabla_Y Z - \nabla_{\nabla_X Y} Z \quad (1)$$

is called the **second covariant derivative** of $Z$ in direction $X, Y$.

**Exercise**

Check that $(\nabla_X (\nabla Z))(Y) = \nabla_X \nabla_Y Z - \nabla_{\nabla_X Y} Z$ actually holds true for all $X, Y, Z \in \mathfrak{X}(M)$. 

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Using **second covariant derivatives of vector fields**, we can write the Riemann curvature tensor as follows:

**Lemma**

The Riemann curvature tensor of a pseudo-Riemannian manifold \((M, g)\) with Levi-Civita connection \(\nabla\) fulfills

\[
R(X, Y)Z = \nabla^2_{X,Y}Z - \nabla^2_{Y,X}Z
\]

for all \(X, Y, Z \in \mathfrak{X}(M)\).

**Proof:**

- **torsion-freeness** of \(\nabla \Rightarrow\)
  
  \(-\nabla_{\nabla_XY}Z + \nabla_{\nabla_YX}Z = -\nabla_{[X,Y]}Z\)

- **writing out** \(\nabla^2_{X,Y}Z - \nabla^2_{Y,X}Z\) with the above proves our claim

**Hence:** The Riemann curvature tensor **describes** “how much” second covariant derivatives are **not symmetric**. In the **flat** case, second covariant derivatives **do commute**.
Remark

Instead of defining the Riemann curvature tensor of \((M, g)\) as a \((1, 3)\)-tensor field, we could have taken the other common approach and define it as a \((0, 4)\)-tensor field \(\tilde{R} \in \mathcal{F}^{0,4}(M)\) given by

\[
\tilde{R}(X, Y, Z, W) := g(R(X, Y)Z, W) \quad \forall X, Y, Z, W \in \mathfrak{X}(M)
\]

It is clear that \(R\) can be recovered from \(\tilde{R}\) by raising the fitting index. In local coordinates \((x^1, \ldots, x^n)\), \(\tilde{R}\) is of the form

\[
\tilde{R} = \sum_{i,j,k,\ell} R_{ij\ell} dx^i \otimes dx^j \otimes dx^k \otimes dx^{\ell},
\]

where \(R_{ij\ell} = \sum_m g_{\ell m} R^m_{ijk}\). 
In Riemannian normal coordinates, the Riemann curvature tensor determines the second order terms in the Taylor expansion of the metric near the reference point:

**Lemma**

Let \((M, g)\) be a pseudo-Riemannian manifold with Levi-Civita connection \(\nabla\) and let \(\varphi = (x^1, \ldots, x^n)\) be **Riemannian normal coordinates** at \(p \in M\) corresponding to a choice of orthonormal basis \(\{v_1, \ldots, v_n\}\) of \(T_p M\). Then the local prefactors \(g_{ij}\) of \(g\) fulfil

\[
\frac{\partial^2 g_{ij}}{\partial x^k \partial x^\ell}(p) = \frac{2}{3} R_{ijk\ell}(p).
\]

for all \(1 \leq i, j, k, \ell \leq n\).

**Proof:** See Prop. 3.1.12 in C. Bär’s *Differential Geometry*, lecture notes (2013).
Next, we will study the so-called **sectional curvature**.

**Definition**

Let \((M, g)\) be a pseudo-Riemannian manifold with Riemann curvature tensor \(R\). Let \(\Pi \subset T_p M\) be a nondegenerate plane spanned by linearly independent vectors \(v, w \in T_p M\). The **sectional curvature** of \(\Pi\) is defined by

\[
K(\Pi) := K(v, w) := \frac{g(R(v, w)w, v)}{g(v, v)g(w, w) - g(v, w)^2}.
\]

We need to check that the sectional curvature is well-defined, i.e. that \(K(\Pi)\) is independent of the basis vectors \(v, w\) of \(\Pi\).
Lemma

\( K \) only depends on the plane \( \Pi \), not on the choice of basis vectors \( v, w \) of \( \Pi \).

Proof:

- let \( \{ V, W \} \) be another basis of \( \Pi \)
- write \( v = aV + bW, \quad w = cV + dW \) for \( a, b, c, d \in \mathbb{R} \)
- since both \( \{ v, w \} \) and \( \{ V, W \} \) are a basis of \( \Pi \), obtain

\[
\begin{vmatrix}
a & b \\
c & d \\
\end{vmatrix} = ad - bc \neq 0
\]

- we check that

\[
g(R(v, w)w, v) = (ad - bc)^2 g(R(V, W)W, V) \quad \text{and}
g(v, v)g(w, w) - g(v, w)^2 =
(ad - bc)^2(g(V, V)g(W, W) - g(V, W)^2) \quad \text{which proves our claim}
\]

- this also proves that \( \Pi \) is nondegenerate if and only if

\[
g(v, v)g(w, w) - g(v, w)^2 \neq 0 \]
Definition
A pseudo-Riemannian manifold \((M, g)\) is of **constant curvature** if its sectional curvatures coincide at every point for every nondegenerate plane in the corresponding tangent space.

Examples
The following pseudo-Riemannian manifolds have **constant curvature**:
- \((\mathbb{R}^n, \langle \cdot, \cdot \rangle_\nu)\) for all \(0 \leq \nu \leq n\)
- \(S^n \subset \mathbb{R}^{n+1}\) equipped with \(g = \langle \cdot, \cdot \rangle|_{T^{S^n} \times T^{S^n}}\) has **positive** constant curvature
- the **hyperbolic upper half plane** \(\{y > 0\} \subset \mathbb{R}^2\) with Riemannian metric \(\frac{dx^2 + dy^2}{y^2}\) has **negative** constant curvature
Question: Can we recover the Riemann curvature tensor from the sectional curvatures?
Answer: Yes, need the following concept:

**Definition**

A *(1, 3)*-tensor

\[ F \in T_p^{1,3} M, \ F : (u, \nu, w) \mapsto F(u, \nu)w \in T_p M \ \forall u, \nu, w \in T_p M, \]

on a pseudo-Riemannian manifold \((M, g)\) is called **abstract curvature tensor** if it fulfils the **identities**

1. \( F(\nu, w) = -F(w, \nu), \)
2. \( g(F(\nu, w)V, W) = -g(V, F(\nu, w)W), \)
3. \( \sum_{\text{cycl.}} F(u, \nu)w = 0 \)

for all \( u, \nu, w, V, W \in T_p M. \)
Lemma C

Let \((M, g)\) be a pseudo-Riemannian manifold with Riemann curvature tensor \(R\) and assume that for \(p \in M\) fixed and an abstract curvature tensor \(F \in T^1_1 M\)

\[
K(v, w) = \frac{g(F(v, w)w, v)}{g(v, v)g(w, w) - g(v, w)^2}
\]

for all linearly independent \(v, w \in T_p M\) spanning a nondegenerate plane in \(T_p M\). Then \(F = R_p\).

Proof: (see right hand side)
As a consequence of Lemma C we obtain:

**Corollary**

Let \((M, g)\) be a pseudo-Riemannian manifold with constant sectional curvature \(K = c \in \mathbb{R}\). Then the Riemann curvature tensor of \((M, g)\) fulfills

\[
R(X, Y)Z = c(g(Y, Z)X - g(X, Z)Y)
\]

for all \(X, Y, Z \in \mathfrak{X}(M)\).

**Proof:**

- Check that for every point \(p \in M\),
- \(c(g(Y, Z)X - g(X, Z)Y)\) restricted to \(T_pM \times T_pM \times T_pM\) defines an abstract curvature tensor, fulfilling
- \(K(v, w) = c\)
- for all \(v, w\) spanning a nondegenerate plane in \(T_pM\)

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Lemma C now implies that
\[ R(X, Y)Z = c(g(Y, Z)X - g(X, Z)Y) \]
holds at \( p \in M \)

since \( p \in M \) was arbitrary this finishes the proof

Next, we will introduce the **Ricci curvature** which is obtained by contracting the Riemann curvature tensor.

**Definition**

Let \((M, g)\) be a pseudo-Riemannian manifold with Riemann curvature tensor \( R \). The **Ricci curvature** \( \text{Ric} \in T^{0,2}(M) \) is defined as

\[ \text{Ric}(X, Y) := \text{tr}(R(\cdot, X)Y) \]

for all \( X, Y \in \mathfrak{X}(M) \) where

\[ R(\cdot, X)Y \in \mathcal{F}^{1,1}(M), \quad R(\cdot, X)Y : Z \mapsto R(Z, X)Y. \]

In **local coordinates** \((x^1, \ldots, x^n)\), \( \text{Ric} \) is of the form

\[
\text{Ric} = \sum_{i,j=1}^{n} \text{Ric}_{ij} dx^i \otimes dx^j = \sum_{i,j=1}^{n} \left( \sum_{k=1}^{n} R^k_{\ ij} \right) dx^i \otimes dx^j.
\]
Exercise

- Show that Ric is symmetric, that is 
  \( \text{Ric}(X, Y) = \text{Ric}(Y, X) \) for all \( X, Y \in \mathfrak{X}(M) \).
- Determine a local formula for each \( \text{Ric}_{ij} \) in terms of the Christoffel symbols.
- Find a formula for Ric for pseudo-Riemannian manifolds of constant curvature.

Note: The Ricci curvature plays a prominent role in general relativity and, as indicated by the name, the study of the Ricci flow.

Definition

In case that \( \text{Ric} = \lambda g \) for a pseudo-Riemannian manifold \( (M, g) \) and some real number \( \lambda \in \mathbb{R} \), \( (M, g) \) is called Einstein manifold.
The Ricci curvature can be used to define a **scalar curvature invariant** as follows:

**Definition**

The **scalar curvature** of a pseudo-Riemannian manifold \((M, g)\) is defined as

\[
S := \text{tr}_g (\text{Ric}) \in C^\infty (M).
\]

**Note:** \(S\) is well defined because of the **symmetry** of \(\text{Ric}\).

In local coordinates \((x^1, \ldots, x^n)\), \(S\) is of the form

\[
S = \sum_{i,j,k,\ell} R^m_{mij} g^{ij} = \sum_{i,j,k,\ell} R_{k\ell ij} g^{k\ell} g^{ij}.
\]

**Exercise**

Find a local formula of the scalar curvature in terms of the **Christoffel symbols**.
In good situations, the scalar curvature can be used to show that two given pseudo-Riemannian manifolds are **not isometric**:

**Lemma**

The **number of isolated local minima and maxima** of the scalar curvature of a pseudo-Riemannian manifold is **invariant under isometries**.

**Proof:**

- let \((M, g)\) and \((N, h)\) be two **isometric** pseudo-Riemannian manifolds with **scalar curvature** \(S_M, S_N\), respectively, and let \(F : M \to N\) be an isometry
- Lemma \(B \leadsto S_N = S_M \circ F\)
- \(F\) is in particular a **diffeomorphism**, hence the claim of this lemma follows
The scalar curvature can also be **calculated** from the **sectional curvatures**:

**Lemma**

Let \((M, g)\) be an \(n \geq 2\)-dimensional pseudo-Riemannian manifold. For \(p \in M\) fixed let \(\{v_1, \ldots, v_n\}\) be an **orthonormal basis** of \(T_p M\). Then

\[
S(p) = \sum_{i \neq j} K(v_i, v_j).
\]

**Proof:** Exercise!

**Remark**

Another commonly studied scalar curvature invariant of pseudo-Riemannian manifolds is the so-called **Kretschmann scalar** which is for a pseudo-Riemannian manifold \((M, g)\) given by \(g(R, R) \in C^\infty(M)\).
END OF LECTURE 18

Next lecture:

- first and second fundamental form
- geodesics & curvature of pseudo-Riemannian submanifolds