Differential geometry
Lecture 18: Curvature

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1. Riemann curvature tensor
2. Sectional curvature
3. Ricci curvature
4. Scalar curvature

Recap of lecture 17:

- defined **geodesics** in pseudo-Riemannian manifolds as curves with parallel velocity
- viewed geodesics as **projections of integral curves** of a vector field $G \in \mathfrak{X}(TM)$ with local flow called **geodesic flow**
- obtained **uniqueness** and **existence** properties of geodesics
- constructed the **exponential map** $\exp : V \to M$, V neighbourhood of the zero-section in $TM \to M$
- showed that geodesics with compact domain are precisely the **critical points** of the **energy functional**
- used the exponential map to construct **Riemannian normal coordinates**, studied local forms of the metric and the Christoffel symbols in such coordinates
- discussed the **Hopf-Rinow Theorem**
- **erratum:** codomain of (x, v) as **local integral curve** of G is $d\varphi(TU)$, not TM

Intuitively, a meaningful definition of the term "**curvature**" for a **smooth surface** in $\mathbb{R}^3$, written locally as a **graph of a smooth function** $f : U \subset \mathbb{R}^2 \to \mathbb{R}$, should involve the **second partial derivatives of** f at each point. How can we find a **coordinate-free** definition of curvature **not just for surfaces** in $\mathbb{R}^3$, which are automatically Riemannian manifolds by restricting $\langle \cdot, \cdot \rangle$, but for **all pseudo-Riemannian manifolds**?

Definition

Let (M, g) be a pseudo-Riemannian manifold with Levi-Civita connection ∇. The **Riemann curvature tensor** of (M, g) is defined as

$$R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M),$$
$$R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

for all $X, Y, Z \in \mathfrak{X}(M)$. In the above formula, we understand $\nabla_X \nabla_Y Z$ as $\nabla_X(\nabla_Y Z)$, analogously for X and Y interchanged.

The first thing we need to check is if R is, as implied in its definition, actually a tensor field:

Lemma

The Riemann curvature tensor is, in fact, a **tensor field**, i.e. $R \in \mathcal{T}^{1,3}(M)$.

Proof: Direct calculation. □

Note: If we would **replace** ∇ with $\mathcal{L}$ in the definition of R, it would identically vanish by the **Jacobi identity**. Also observe that the Riemann curvature tensor **vanishes identically if** $\dim(M) = 1$.

Question: Why should we study the Riemann curvature tensor R in the first place? What is the **geometric picture** one should have in mind for R?

(partial) Answer: (see next page)

Lemma A

Let (M,g) be a pseudo-Riemannian manifold with Levi-Civita connection ∇. For any $X \in \mathfrak{X}(M)$, denote for every $p \in M$ by $P_0^t(X) : T_pM \to T_{\gamma(t)}M$ the **parallel transport map** with respect to ∇ **along the integral curve** $\gamma : (-\varepsilon,\varepsilon) \to M$ of X with $\gamma(0) = p$ for $\varepsilon > 0$ small enough, that is for $p \in M$ fixed we have $P_0^t(X) = P_0^t(\gamma)$. Let $(x^1,\dots,x^n)$ be local coordinates on $U \subset M$. Then

$$R\left(\left.\frac{\partial}{\partial x^i}\right|_p, \left.\frac{\partial}{\partial x^j}\right|_p\right)\left.\frac{\partial}{\partial x^k}\right|_p =$$

$$\left.\frac{\partial}{\partial s}\right|_{s=0}\left.\frac{\partial}{\partial t}\right|_{t=0} P_0^s\left(\frac{\partial}{\partial x^i}\right)^{-1} P_0^t\left(\frac{\partial}{\partial x^j}\right)^{-1} P_0^s\left(\frac{\partial}{\partial x^i}\right) P_0^t\left(\frac{\partial}{\partial x^j}\right)\left.\frac{\partial}{\partial x^k}\right|_p$$

for all $1 \leq i,j,k \leq n$ and all $p \in U$. The Riemann curvature tensor is the **unique** $(1,3)$-tensor field fulfilling the above equation in all local coordinates.

Proof: (next page)

(continuation of proof)

- in **coordinate representations**, $P_0^t\left(\frac{\partial}{\partial x^j}\right) : T_pM \to T_{\gamma(t)}M$, $t \in (-\varepsilon, \varepsilon)$, and the other parallel translations are **smooth maps** of the form

$$\widehat{P_0^t\left(\frac{\partial}{\partial x^j}\right)} : (-\varepsilon, \varepsilon) \to \mathrm{GL}(n),$$

  where $\mathrm{GL}(n)$ **being the codomain** follows the fact that parallel translations are **isometries**, hence **isomorphisms**, for each fixed t

- the above map should be understood as mapping **prefactors** of vectors in T_pM written in the **coordinate basis** to **prefactors** of vectors in $T_{\gamma(t)}M$, again written in the **coordinate basis**

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(continuation of proof)

- hence, the **partial derivatives of products** of $\widehat{P_0^t\left(\frac{\partial}{\partial x^j}\right)}$ behave according to the **product rule of matrix valued curves**, i.e. **for all** $A, B : (-\varepsilon, \varepsilon) \to \mathrm{GL}(n)$ **smooth with** $A(0) = B(0) = \mathbb{1}$ and all $v \in \mathbb{R}^n$ we have

$$
\begin{aligned}
&\frac{\partial}{\partial s}\Big|_{s=0} \frac{\partial}{\partial t}\Big|_{t=0} A(s)^{-1}B(t)^{-1}A(s)B(t)v \\
&= \left(\frac{\partial}{\partial s}\Big|_{s=0} A(s)^{-1}\right)\left(\frac{\partial}{\partial t}\Big|_{t=0} B(t)^{-1}\right)v \\
&+ \left(\frac{\partial}{\partial t}\Big|_{t=0} B(t)^{-1}\right)\left(\frac{\partial}{\partial s}\Big|_{s=0} A(s)\right)v \\
&= \left(\frac{\partial}{\partial s}\Big|_{s=0} A(s)\right)\left(\frac{\partial}{\partial t}\Big|_{t=0} B(t)\right)v \\
&- \left(\frac{\partial}{\partial t}\Big|_{t=0} B(t)\right)\left(\frac{\partial}{\partial s}\Big|_{s=0} A(s)\right)v
\end{aligned}
$$

- note that $\frac{\partial}{\partial s}\Big|_{s=0} A(s) \in \mathrm{End}(\mathbb{R}^n)$, meaning that the derivative is in general **not invertible**

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(continuation of proof)

- hence, using our formula **relating ∇ with parallel transport maps**, obtain

$$\frac{\partial}{\partial s}\Big|_0 \frac{\partial}{\partial t}\Big|_0 P_0^s\left(\frac{\partial}{\partial x^i}\right)^{-1} P_0^t\left(\frac{\partial}{\partial x^j}\right)^{-1} P_0^s\left(\frac{\partial}{\partial x^i}\right) P_0^t\left(\frac{\partial}{\partial x^j}\right) \frac{\partial}{\partial x^k}\Big|_p$$
$$= \left(\nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k} - \nabla_{\frac{\partial}{\partial x^j}} \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^k}\right)\Bigg|_p$$

  proving the **first statement** of this lemma

- in order to show that R is indeed the **unique tensor field fulfilling the above**, we only need to check that for any local functions $X^1, Y^1, Z^1, \ldots, X^n, Y^n, Z^n \in C^\infty(U)$ and all $p \in U$, $\sum_{i,j,k} X^i(p) Y^j(p) Z^k(p) R\left(\frac{\partial}{\partial x^i}\Big|_p, \frac{\partial}{\partial x^j}\Big|_p\right) \frac{\partial}{\partial x^k}\Big|_p$ and $(R(X,Y)Z)|_p$ via its initial definition **coincide** **[**Exercise!**]** □

We have the following **local formula** for the Riemann curvature tensor:

Lemma

In **local coordinates** $(x^1, \dots, x^n)$ the Riemann curvature tensor of a pseudo-Riemannian manifold (M, g) has components

$$R^{\ell}{}_{ijk} := dx^{\ell}\left(R\left(\tfrac{\partial}{\partial x^i}, \tfrac{\partial}{\partial x^j}\right)\tfrac{\partial}{\partial x^k}\right),$$

so that **locally** $R = \sum_{i,j,k,\ell} R^{\ell}{}_{ijk} \frac{\partial}{\partial x^{\ell}} \otimes dx^i \otimes dx^j \otimes dx^k$. The local functions $R^{\ell}{}_{ijk}$ are given by

$$R^{\ell}{}_{ijk} = \frac{\partial \Gamma^{\ell}_{jk}}{\partial x^i} - \frac{\partial \Gamma^{\ell}_{ik}}{\partial x^j} + \sum_{m=1}^{n}\left(\Gamma^{\ell}_{im}\Gamma^{m}_{jk} - \Gamma^{\ell}_{jm}\Gamma^{m}_{ik}\right)$$

for all $1 \leq i, j, k, \ell \leq n$.

Proof: Direct calculation. □

The Riemann curvature tensor R of a pseudo-Riemannian manifold (M, g) fulfils the following **identities**:

Lemma

- i $R(X,Y) = -R(Y,X)$,
- ii $g(R(X,Y)Z, W) = -g(Z, R(X,Y)W)$,
- iii $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$ (**first** or **algebraic Bianchi identity**),
- iv $g(R(X,Y)Z, W) = g(R(Z,W)X, Y)$,
- v $(\nabla_X R)(Y,Z) + (\nabla_Y R)(Z,X) + (\nabla_Z R)(X,Y) = 0$ (**second** or **differential Bianchi identity**)

for all $X, Y, Z, W \in \mathfrak{X}(M)$.

Proof: (see notes on the right hand side)

As one might expect from the **tensoriality**, the Riemann curvature tensor behaves well under **isometries**.

Lemma B

Let $F : (M, g) \to (N, h)$ be an **isometry** and let R^M and R^N denote the **Riemann curvature tensors** of (M, g) and (N, h), respectively. Then

$$F_*(R^M(X, Y)Z) = R^N(F_*X, F_*Y)F_*Z$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

Proof:

- **suffices** to show that $F_*\nabla^M_X Y = \nabla^N_{F_*X}(F_*Y)$ for all $X, Y \in \mathfrak{X}(M)$, where ∇^M and ∇^N denote the Levi-Civita connections of (M, g) and (N, h), respectively
- $\rightsquigarrow$ use **Koszul formula** for ∇^M and ∇^N, **bijectivity** of $F_* : \mathfrak{X}(M) \to \mathfrak{X}(N)$, and $F_*[X, Y] = [F_*X, F_*Y]$ $\square$

Definition

A pseudo-Riemannian manifold with **vanishing Riemann curvature tensor** is called **flat**.

Examples

The following pseudo-Riemannian manifolds are **flat**:

- $(\mathbb{R}^n, \langle\cdot,\cdot\rangle_\nu)$, $0 \leq \nu \leq n$
- the **cylinder** $\mathbb{R} \times S^1$ and the **2-torus** $T^2 = S^1 \times S^1$ equipped with the respective **product metric**
- more generally, $(M \times N, g + h)$ for all (M, g) **and** (N, h) **flat**

Suppose that we are given a **flat Riemannian** manifold (M, g).
Question: Is (M, g) automatically of a **simple form**, at least locally (up to isometry)?
Answer: Yes! **Locally**, we have the following result:

Theorem

An n-dimensional **Riemannian manifold** (M, g) is **flat** if and only if it is **locally isometric** to $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$, meaning that for all $p \in M$ there exists an open neighbourhood $U \subset M$ of p and an isometry $F : (U, g) \to (F(U), \langle \cdot, \cdot \rangle)$, $F(U) \subset \mathbb{R}^n$ open.

Proof:

- Lemma B $\rightsquigarrow$ **local isometry** to $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ **implies flatness**, i.e. $R \equiv 0$
- the other direction of this proof requires **a lot** more work, for details see Theorem 7.3 in J.M. Lee's *Riemannian Manifolds – An Introduction to Curvature*, Springer GTM 176 (1997) (with slightly different conventions)

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(continuation of proof)

- the idea is to construct a **commuting orthonormal local frame** of $TM \to M$ near every given point
- the key ingredient is that **parallel transport of vectors** at, say, $p \in M$, to a close enough point $q \in M$ **does not depend on the chosen curve** starting at p and ending at q if it is required to be contained in a small enough open neighbourhood of both p and q
- follows from a **similar argument** as in Lemma A □

In Lemma A we have described how to interpret the Riemann curvature tensor **geometrically** as **infinitesimal change of parallel transport** of tangent vectors **around infinitesimal parallelograms**.

Question: Is there **another motivation** for the definition of the Riemann curvature tensor?

Answer: Yes, via **second covariant derivatives**!

Definition

Let (M, g) be a **pseudo-Riemannian manifold** with **Levi-Civita connection** ∇. Then for all $X, Y, Z \in \mathfrak{X}(M)$,

$$\nabla^2_{X,Y} Z := (\nabla_X(\nabla Z))(Y) = \nabla_X \nabla_Y Z - \nabla_{\nabla_X Y} Z \qquad (1)$$

is called the **second covariant derivative** of Z in direction X, Y.

Exercise

Check that $(\nabla_X(\nabla Z))(Y) = \nabla_X \nabla_Y Z - \nabla_{\nabla_X Y} Z$ actually **holds true** for all $X, Y, Z \in \mathfrak{X}(M)$.

Using **second covariant derivatives of vector fields**, we can write the Riemann curvature tensor as follows:

Lemma

The **Riemann curvature tensor** of a pseudo-Riemannian manifold (M, g) with Levi-Civita connection ∇ **fulfils**

$$R(X, Y)Z = \nabla^2_{X,Y} Z - \nabla^2_{Y,X} Z \tag{2}$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

Proof:

- **torsion-freeness** of $\nabla \rightsquigarrow$ $-\nabla_{\nabla_X Y} Z + \nabla_{\nabla_Y X} Z = -\nabla_{[X,Y]} Z$
- **writing out** $\nabla^2_{X,Y} Z - \nabla^2_{Y,X} Z$ with the above proves our claim □

Hence: The Riemann curvature tensor **describes** "how much" second covariant derivatives are **not symmetric**. In the **flat** case, second covariant derivatives **do commute**.

Remark

Instead of defining the Riemann curvature tensor of (M, g) as a $(1,3)$-**tensor field**, we could have taken the other **common approach** and define it as a $(0,4)$-**tensor field** $\widetilde{R} \in \mathcal{T}^{0,4}(M)$ given by

$$\widetilde{R}(X, Y, Z, W) := g(R(X, Y)Z, W) \quad \forall X, Y, Z, W \in \mathfrak{X}(M)$$

It is clear that R **can be recovered from** $\widetilde{R}$ by raising the fitting index. In local coordinates $(x^1, \dots, x^n)$, $\widetilde{R}$ is of the form

$$\widetilde{R} = \sum_{i,j,k,\ell} R_{ijk\ell}\, dx^i \otimes dx^j \otimes dx^k \otimes dx^\ell,$$

where $R_{ijk\ell} = \sum_m g_{\ell m} R^m{}_{ijk}$.

In **Riemannian normal coordinates**, the Riemann curvature tensor determines the **second order terms** in the **Taylor expansion** of the metric near the reference point:

Lemma

Let (M, g) be a pseudo-Riemannian manifold with Levi-Civita connection ∇ and let $\varphi = (x^1, \dots, x^n)$ be **Riemannian normal coordinates** at $p \in M$ corresponding to a choice of **orthonormal basis** $\{v_1, \dots, v_n\}$ of T_pM. Then the local prefactors g_{ij} of g fulfil

$$\frac{\partial^2 g_{ij}}{\partial x^k \partial x^\ell}(p) = \frac{2}{3} R_{ijk\ell}(p).$$

for all $1 \leq i, j, k, \ell \leq n$.

Proof: See Prop. 3.1.12 in C. Bär's *Differential Geometry*, lecture notes (2013). □

Next, we will study the so-called **sectional curvature**.

Definition

Let (M, g) be a pseudo-Riemannian manifold with Riemann curvature tensor R. Let $\Pi \subset T_pM$ be a **nondegenerate plane** spanned by linearly independent vectors $v, w \in T_pM$. The **sectional curvature** of Π is defined by

$$K(\Pi) := K(v, w) := \frac{g(R(v, w)w, v)}{g(v, v)g(w, w) - g(v, w)^2}.$$

$\rightsquigarrow$ need to check that the sectional curvature is **well-defined**, i.e. that $K(\Pi)$ is **independent of the basis vectors** v, w of Π

Lemma

K **only depends on the plane** Π, not on the choice of basis vectors v, w of Π.

Proof:

- let $\{V, W\}$ be **another basis** of Π
- write $v = aV + bW$, $w = cV + dW$ for $a, b, c, d \in \mathbb{R}$
- since **both** $\{v, w\}$ and $\{V, W\}$ **are a basis** of Π, obtain

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc \neq 0$$

- we check that $g(R(v, w)w, v) = (ad - bc)^2 g(R(V, W)W, V)$ and $g(v, v)g(w, w) - g(v, w)^2 = (ad - bc)^2 (g(V, V)g(W, W) - g(V, W)^2)$ which **proves our claim**
- this **also proves** that Π is **nondegenerate if and only if** $g(v, v)g(w, w) - g(v, w)^2 \neq 0$ $\square$

Definition

A pseudo-Riemannian manifold (M, g) is of **constant curvature** if its **sectional curvatures coincide at every point for every nondegenerate plane** in the corresponding tangent space.

Examples

The following pseudo-Riemannian manifolds have **constant curvature**:

- $(\mathbb{R}^n, \langle\cdot,\cdot\rangle_\nu)$ for all $0 \leq \nu \leq n$
- $S^n \subset \mathbb{R}^{n+1}$ equipped with $g = \langle\cdot,\cdot\rangle|_{TS^n \times TS^n}$ has **positive** constant curvature
- the **hyperbolic upper half plane** $\{y > 0\} \subset \mathbb{R}^2$ with Riemannian metric $\frac{dx^2+dy^2}{y^2}$ has **negative** constant curvature

Question: Can we **recover** the Riemann curvature tensor from the sectional curvatures?
Answer: Yes, need the following concept:

Definition

A $(1,3)$-**tensor**

$$F \in T_p^{1,3}M,\ F : (u,v,w) \mapsto F(u,v)w \in T_pM\ \forall u,v,w \in T_pM,$$

on a pseudo-Riemannian manifold (M,g) is called **abstract curvature tensor** if it fulfils the **identities**

1. $F(v,w) = -F(w,v)$,
2. $g(F(v,w)V,W) = -g(V,F(v,w)W)$,
3. $\sum_{\text{cycl.}} F(u,v)w = 0$

for all $u,v,w,V,W \in T_pM$.

Lemma C

Let (M, g) be a pseudo-Riemannian manifold with Riemann curvature tensor R and assume that **for** $p \in M$ **fixed** and an **abstract curvature tensor** $F \in T_p^{1,3}M$

$$K(v, w) = \frac{g(F(v, w)w, v)}{g(v, v)g(w, w) - g(v, w)^2}$$

for all linearly independent $v, w \in T_pM$ spanning a **nondegenerate plane** in T_pM. Then $F = R_p$.

Proof: (see right hand side)

As a **consequence** of Lemma C we obtain:

Corollary

Let (M, g) be a pseudo-Riemannian manifold with **constant sectional curvature** $K = c \in \mathbb{R}$. Then the **Riemann curvature tensor** of (M, g) **fulfils**

$$R(X, Y)Z = c(g(Y, Z)X - g(X, Z)Y)$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

Proof:

- $\rightsquigarrow$ check that for every point $p \in M$, $c(g(Y, Z)X - g(X, Z)Y)$ restricted to $T_pM \times T_pM \times T_pM$ **defines an abstract curvature tensor**, fulfilling

$$K(v, w) = c$$

  for all v, w spanning a **nondegenerate plane** in T_pM

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(continuation of proof)

- Lemma C now implies that $R(X,Y)Z = c(g(Y,Z)X - g(X,Z)Y)$ holds **at** $p \in M$
- since $p \in M$ was **arbitrary** this finishes the proof □

Next, we will introduce the **Ricci curvature** which is obtained by **contracting the Riemann curvature tensor**.

Definition

Let (M,g) be a pseudo-Riemannian manifold with Riemann curvature tensor R. The **Ricci curvature** $\mathrm{Ric} \in \mathcal{T}^{0,2}(M)$ is defined as

$$\mathrm{Ric}(X,Y) := \mathrm{tr}(R(\cdot,X)Y)$$

for all $X, Y \in \mathfrak{X}(M)$ where

$$R(\cdot,X)Y \in \mathcal{T}^{1,1}(M), \quad R(\cdot,X)Y : Z \mapsto R(Z,X)Y.$$

In **local coordinates** $(x^1,\ldots,x^n)$, Ric is of the form

$$\mathrm{Ric} = \sum_{i,j=1}^{n} \mathrm{Ric}_{ij}\, dx^i \otimes dx^j = \sum_{i,j=1}^{n} \left(\sum_{k=1}^{n} R^k{}_{kij} \right) dx^i \otimes dx^j.$$

Exercise

- Show that Ric is **symmetric**, that is $\mathrm{Ric}(X, Y) = \mathrm{Ric}(Y, X)$ for all $X, Y \in \mathfrak{X}(M)$.
- Determine a **local formula** for each Ric_{ij} in terms of the **Christoffel symbols**.
- Find a formula for Ric for pseudo-Riemannian manifolds of **constant curvature**.

Note: The Ricci curvature plays a prominent role in **general relativity** and, as indicated by the name, the study of the **Ricci flow**.

Definition

In case that $\mathrm{Ric} = \lambda g$ for a pseudo-Riemannian manifold (M, g) and some real number $\lambda \in \mathbb{R}$, (M, g) is called **Einstein manifold**.

The Ricci curvature can be used to define a **scalar curvature invariant** as follows:

Definition

The **scalar curvature** of a pseudo-Riemannian manifold (M, g) is defined as

$$S := \mathrm{tr}_g(\mathrm{Ric}) \in C^\infty(M).$$

Note: S is well defined because of the **symmetry** of Ric.
In local coordinates $(x^1, \ldots, x^n)$, S is of the form

$$S = \sum_{i,j,k,\ell} R^m{}_{mij}\, g^{ij} = \sum_{i,j,k,\ell} R_{k\ell ij}\, g^{k\ell} g^{ij}.$$

Exercise

Find a local formula of the scalar curvature in terms of the **Christoffel symbols**.

In good situations, the scalar curvature can be used to show that two given pseudo-Riemannian manifolds are **not isometric:**

Lemma

The **number of isolated local minima and maxima** of the scalar curvature of a pseudo-Riemannian manifold is **invariant under isometries**.

Proof:

- let (M, g) and (N, h) be two **isometric** pseudo-Riemannian manifolds with **scalar curvature** S_M, S_N, respectively, and let $F : M \to N$ be an isometry
- Lemma $B \rightsquigarrow S_N = S_M \circ F$
- F is in particular a **diffeomorphism**, hence the claim of this lemma follows □

The scalar curvature can also be **calculated** from the **sectional curvatures**:

Lemma

Let (M, g) be an $n \geq 2$-dimensional pseudo-Riemannian manifold. For $p \in M$ fixed let $\{v_1, \ldots, v_n\}$ be an **orthonormal basis of** T_pM. Then

$$S(p) = \sum_{i \neq j} K(v_i, v_j).$$

Proof: Exercise!

Remark

Another commonly studied scalar curvature invariant of pseudo-Riemannian manifolds is the so-called **Kretschmann scalar** which is for a pseudo-Riemannian manifold (M, g) given by $g(R, R) \in C^\infty(M)$.

END OF LECTURE 18

Next lecture:

- first and second fundamental form
- geodesics & curvature of pseudo-Riemannian submanifolds