# Differential geometry
# Lecture 17: Geodesics and the exponential map

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**Recap of lecture 16:**

- constructed **covariant derivatives along curves**
- defined **parallel transport**
- studied the **relation** between a given connection in the tangent bundle and its parallel transport maps
- introduced **torsion tensor** and **metric connections**, studied geometric interpretation
- defined the **Levi-Civita connection** of a pseudo-Riemannian manifold

Recall the definition of the acceleration of a smooth curve $\gamma : I \to \mathbb{R}^n$, that is $\gamma'' \in \Gamma_\gamma(T\mathbb{R}^n)$.
**Question:** Is there a **coordinate-free** analogue of this construction involving connections?
**Answer:** Yes, uses **covariant derivative along curves**.

## Definition

Let $M$ be a smooth manifold, $\nabla$ a connection in $TM \to M$, and $\gamma : I \to M$ a smooth curve. Then $\nabla_{\gamma'}\gamma' \in \Gamma_\gamma(TM)$ is called the **acceleration** of $\gamma$ (with respect to $\nabla$).

Of particular interest is the case if the acceleration of a curve **vanishes**, that is if its **velocity vector field is parallel**:

## Definition

A smooth curve $\gamma : I \to M$ is called **geodesic** with respect to a given connection $\nabla$ in $TM \to M$ if $\nabla_{\gamma'}\gamma' = 0$.

In local coordinates $(x^1, \ldots, x^n)$ on $M$ we obtain for any smooth curve $\gamma : I \to M$ the **local formula**

$$\nabla_{\gamma'}\gamma' = \sum_{k=1}^{n} \left( \frac{\partial^2 \gamma^k}{\partial t^2} + \sum_{i,j=1}^{n} \frac{\partial \gamma^i}{\partial t} \frac{\partial \gamma^j}{\partial t} \Gamma_{ij}^k \right) \frac{\partial}{\partial x^k}.$$

As a consequence we obtain a **local form of the geodesic equation**:

## Corollary

$\gamma$ is a geodesic **if and only if** in all local coordinates covering a nontrivial subset of the image of $\gamma$ it holds that

$$\frac{\partial^2 \gamma^k}{\partial t^2} + \sum_{i,j=1}^{n} \frac{\partial \gamma^i}{\partial t} \frac{\partial \gamma^j}{\partial t} \Gamma_{ij}^k = 0 \quad \forall 1 \leq k \leq n,$$

where one usually writes $\Gamma_{ij}^k$ instead of $\Gamma_{ij}^k \circ \gamma$.

**Alternative notation:** $\ddot{x}^k + \dot{x}^i \dot{x}^j \Gamma_{ij}^k = 0 \ \forall 1 \leq k \leq n$.

## Example

Geodesics w.r.t. the Levi-Civita connection $\nabla$ of $\left(\mathbb{R}^n, \sum_{i=1}^{n}(du^i)^2\right)$ are **affine lines** of constant speed. For any **arbitrary** curve $\gamma : I \to \mathbb{R}^n$ we have

$$\nabla_{\gamma'}\gamma' = \gamma'' = \left(\gamma, \frac{\partial^2\gamma^1}{\partial t^2}, \dots, \frac{\partial^2\gamma^n}{\partial t^2}\right).$$

If a curve is a geodesic with respect to a **metric connection**, e.g. the Levi-Civita connection, it automatically has the following property:

## Lemma

Let $\gamma : I \to M$ be a geodesic on a pseudo-Riemannian manifold $(M, g)$ with respect to a metric connection $\nabla$. Then $g(\gamma', \gamma') : I \to \mathbb{R}$ is constant.

**Proof:** $\frac{\partial(g(\gamma',\gamma'))}{\partial t} = \nabla_{\gamma'}(g(\gamma', \gamma')) = (\nabla_{\gamma'}g)(\gamma', \gamma') + 2g(\nabla_{\gamma'}\gamma', \gamma') = 0$ □

## Corollary

A geodesic $\gamma$ in w.r.t. the Levi-Civita connection of a **Riemannian manifold** $(M, g)$ has **constant speed** $\sqrt{g(\gamma', \gamma')}$.

Next, we need to study if geodesics **always exist** and determine their **uniqueness**.

## Proposition A

- Let $M$ be a smooth manifold and $\nabla$ a connection in $TM \to M$. Let further $p \in M$ and $v \in T_pM$. Then there exists $\varepsilon > 0$ and a smooth curve $\gamma : (-\varepsilon, \varepsilon) \to M$, $\gamma(0) = p$, $\gamma'(0) = v$, such that $\gamma$ is a **geodesic**.
- If $\gamma_1 : I_1 \to M$ and $\gamma_2 : I_2 \to M$ are geodesics on $M$ such that $I_1 \cap I_2 \neq \emptyset$ and for some point $t_0 \in I_1 \cap I_2$, $\gamma_1(t_0) = \gamma_2(t_0)$ and $\gamma_1'(t_0) = \gamma_2'(t_0)$, then $\gamma_1$ and $\gamma_2$ **coincide on** $I_1 \cap I_2$, i.e. $\gamma_1|_{I_1 \cap I_2} = \gamma_2|_{I_1 \cap I_2}$.

**Proof:** (next page)

(continuation of proof)

- suffices to prove this proposition in **local coordinates**
- the **differential equation** for a geodesic in local coordinates $\ddot{x}^k + \dot{x}^i \dot{x}^j \Gamma_{ij}^k = 0$, $1 \leq k \leq n$, is a nonlinear system of **second order ODEs**
- $\rightsquigarrow$ turn this system of $n$ **second order ODEs** into a system of $2n$ **first order system of ODEs**
- $\rightsquigarrow$ system of equations

$$\dot{x}^k = v^k, \ \dot{v}^i = -\dot{x}^i \dot{x}^j \Gamma_{ij}^k \quad \forall 1 \leq k \leq n$$

  with **fitting initial values**
- first thing in need of **clarification**: the symbols $v^k$
- the $v^k$ are precisely the **induced coordinates** on $TU \subset TM$, so that $v^k(V) = V(x^k)$ for all $V \in T_q M$ with $q \in U$
- in the above eqn., the $x^k$ and $v^k$ are, however, to be read as **components of a curve** $(x = x(t), v = v(t)) : I \to TM$

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(continuation of proof)

- in local coordinates $x^k$, $v^k$, the loc. geod. eqn. can be viewed as **integral curve of a vector field on $TU$**, $G \in \mathfrak{X}(TU)$, that is a smooth section in $TTU \to TU$
- $\rightsquigarrow$ to see this first observe that since the $x^k$ and $v^k$ are **coordinate functions on $TU$**, they **induce coordinates on $TTU$**
- the corresponding **local frame in $TTU \to TU$** is given by

$$\left\{ \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial v^1}, \dots, \frac{\partial}{\partial v^n} \right\}$$

- one can imagine each $\frac{\partial}{\partial x^k}$ as being **"horizontal"** and each $\frac{\partial}{\partial v^k}$ as being **"vertical"**
- using **Einstein summation convention**, $G$ is given by

$$G = v^k \frac{\partial}{\partial x^k} - v^i v^j \Gamma_{ij}^k \frac{\partial}{\partial v^k}$$

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(continuation of proof)

- by covering $M$ with charts and thus $TM$ with induced charts, $G$ extends to a **vector field on** $TM$**,** $G \in \mathfrak{X}(TM)$
- since any **integral curve** of $G$, $(x, v) : I \to TM$, $t \mapsto (x(t), v(t))$, fulfils $\dot{x} = v$, it is precisely the **velocity vector field** of the curve $x : I \to M$, $t \mapsto x(t)$
- hence, the **projection** of any integral curve of $G$ to $M$ via the bundle projection $\pi : TM \to M$ is a **geodesic**
- by **existence** and **uniqueness** properties of **integral curves of vector fields**, the statement of this proposition follows □

### Definition

The (local) flow of the vector field $G \in \mathfrak{X}(TM)$, locally given by $G = v^k \frac{\partial}{\partial x^k} - v^i v^j \Gamma^k_{ij} \frac{\partial}{\partial v^k}$ as in Proposition A, is called **geodesic flow** with respect to $\nabla$.

**Local uniqueness** of geodesics allows us to define a **maximality property** for geodesics:

## Definition

A geodesic $\gamma : I \to M$ is called **maximal** if there exists no strictly larger interval $\widetilde{I} \supset I$ and a geodesic $\widetilde{\gamma} : \widetilde{I} \to M$, such that $\widetilde{\gamma}|_I = \gamma$. This means that $\gamma$ cannot be extended to a larger domain while still keeping its **geodesic property**. A smooth manifold with connection $\nabla$ in $TM \to M$ is called **geodesically complete** if every maximal geodesic is defined on $I = \mathbb{R}$. A pseudo-Riemannian manifold $(M, g)$, respectively the metric $g$, is called **geodesically complete** if its Levi-Civita connection is complete.

**Question:** Which **reparametrisation** of geodesics are allowed so that the result is still a geodesic?
**Answer:**

**Lemma**

Let $\gamma : I \to M$ be a geodesic with **non-vanishing speed** with respect to $\nabla$ and $f : I \to I'$ a diffeomorphism. Then $\gamma \circ f$ is a geodesic with non-vanishing speed **if and only if** $f$ is **affine-linear**, that is of the form $f(t) = at + b$ for $a \in \mathbb{R} \setminus \{0\}$, $b \in \mathbb{R}$.

**Proof:**

- **local formula & chain rule** $\rightsquigarrow$

  $$\nabla_{(\gamma \circ f)'}(\gamma \circ f)' = f'' \cdot \gamma' \circ f + (f')^2 \cdot (\nabla_{\gamma'}\gamma') \circ f = f'' \cdot \gamma' \circ f,$$

  where the last equality comes from assumption that $\gamma$ is a **geodesic**
- hence, $\gamma \circ f$ is a geodesic **if and only if** $f'' = 0$, that is if $f = at + b$ with $a \in \mathbb{R} \setminus \{0\}$, $b \in \mathbb{R}$ □

An immediate consequence is:

**Corollary A**

Maximal geodesics are **unique up to affine reparametrisation**. If $\gamma : I \to M$ is a geodesic with initial value $\gamma(0) = p$, $\gamma'(0) = v \in T_pM$, $t \mapsto \gamma_a(t) := \gamma(at)$ is a geodesic with initial value $\gamma_a(0) = p$, $\gamma_a'(t) = av$ for all $a \in \mathbb{R}$. If $a = 0$, the domain of $\gamma_a$ is $\mathbb{R}$. If $a \neq 0$, the domain of $\gamma_a$ is $a^{-1} \cdot I$.

In the case of **pseudo-Riemannian manifolds**, we have the following additional result:

**Corollary**

A geodesic in a pseudo-Riemannian manifold with respect to the Levi-Civita connection with **nonvanishing velocity** can always be parametrised to be of **unit speed**, that is either $g(\gamma', \gamma') \equiv 1$ or $g(\gamma', \gamma') \equiv -1$.

## Example

- Each maximal geodesics of $\mathbb{R}^n$ equipped with the **canonical connection** with initial condition $\gamma(0) = p \in \mathbb{R}^n$, $\gamma'(0) = v \in T_p\mathbb{R}^n \cong \mathbb{R}^n$, is of the form

  $$\gamma : \mathbb{R} \to \mathbb{R}^n, \quad t \mapsto p + tv.$$

  This in particular means that the canonical connection on $\mathbb{R}^n$ is **geodesically complete**.
- Consider $S^n \subset \mathbb{R}^{n+1}$ with **induced metric** $g = \langle \cdot, \cdot \rangle|_{TS^n \times TS^n}$, where $\langle \cdot, \cdot \rangle$ denotes the standard Riemannian metric on $\mathbb{R}^n$. The maximal geodesics of $(S^n, g)$ with respect to the Levi-Civita connection are **great circles**, that is

  $$\gamma : \mathbb{R} \to S^n, \quad t \mapsto e^{At}p$$

  for $\gamma(0) = p \in S^n$, $\gamma'(0) = Ap$, $A \in \mathrm{Mat}(n \times n)$ skew. This, again, means that the Levi-Civita connection of $(S^n, g)$ is **geodesically complete**.

Next, we will construct the **exponential map** of a given connection in the tangent bundle of a smooth manifold. This requires some **technical tools** and **auxiliary results**.

## Definition

Let $M$ be a smooth manifold with connection $\nabla$ in $TM \to M$. An **open neighbourhood of the zero section** in $TM \to M$ is an open set $V \subset TM$ such that for all $p \in M$, $V_p := T_pM \cap V$ is an **open neighbourhood of the origin** $0 \in T_pM$. Note that the **smooth manifold structure** and **topology** on $T_pM$ are induced by the local trivialisations of $TM \to M$ and the corresponding **fibrewise isomorphisms** $T_pM \cong \mathbb{R}^n$.

## Lemma

Let $\nabla$ be a connection in $TM \to M$. Then there **exists an open nbhd. of the zero section** $V \subset TM$, such that for all $v \in V_p \subset V$, the geodesic $\gamma_v$ with initial condition $\gamma_v(0) = p$, $\gamma_v'(0) = v$, has domain **containing the compact interval** $[0, 1]$.

**Proof:** (next page)

(continuation of proof)

- Corollary A implies that if $\gamma_v$ is defined on at least $[0,1]$, $\gamma_{rv}$ for $r \in [0,1]$ is **also defined on** $[0,1]$
- construction in proof of Proposition A $\rightsquigarrow$ geodesics can be viewed as **projections of integral curves** of a vector field on $TM$
- hence, by **identifying** $M$ with the **image of the zero section** in $TM$ and using Corollary A, it follows that in order to prove this proposition it **suffices** to show that for all $p \in M \subset TM$ we can find $\varepsilon_p > 0$ and an open neighbourhood $W_p$ of $p$ in $TM$ (**not** a subset of the fibre $T_pM$), such that all **integral curves** of $G = v^k \frac{\partial}{\partial x^k} - v^i v^j \Gamma^k_{ij} \frac{\partial}{\partial v^k}$ starting in $W_p$ are **defined on at least** $[0, \varepsilon_p]$
- this follows from the fact that $G$ is a **smooth vector field**
- if $\varepsilon_p < 1$, we **rescale** $W_p$ **fibrewise** with **scaling factor** $\varepsilon_p$, so that we can assume w.l.o.g. that all **integral curves of** $G$ **starting in** $W_p$ are defined on **at least** $[0,1]$

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(continuation of proof)

- **repeating this procedure** for all $p \in M \subset TM$, we **obtain our desired open neighbourhood** $V \subset TM$ of the zero section in $TM \to M$ by setting

$$V := \bigcup_{p \in M} W_p$$

$\square$

Now we have all tools at hand to define the exponential map:

**Definition**

Let $V \subset TM$ be an **open neighbourhood of the zero section** in $TM \to M$ such that for all $v \in V$, the **unique maximal geodesic** $\gamma_v$ with respect to $\nabla$ with initial condition $\gamma_v(0) = p$, $\gamma_v'(0) = v$, is **defined on** $[0, 1]$. The **exponential map** with respect to $\nabla$ is defined as

$$\exp : V \to M, \quad v \mapsto \gamma_v(1).$$

The **exponential map at** $p \in M$ $\exp_p : V_p \to M$ is the restriction of exp to $V_p$.

We want to use the exponential map to construct certain **nice local coordinates**. In order to do so, we need:

**Proposition B**

Let $M$ be a smooth manifold and $\nabla$ a connection in $TM \to M$. For all $p \in M$, the exponential map at $p$ is a **local diffeomorphism near** $0 \in T_pM$.

**Proof:**

- suffices to show $d\exp_p = id_{T_pM}$, which together with theorem about **local invertibility** will complete the proof
- let $\gamma_v$ denote the **maximal geodesic with chosen initial value** $\gamma_v(0) = p$, $\gamma_v'(0) = v$ for $v \in T_pM$
- Corollary A implies $d\exp_p(v) = \frac{\partial}{\partial t}\big|_{t=0} \exp_p(tv) = \frac{\partial}{\partial t}\big|_{t=0} \gamma_{tv}(1) = \frac{\partial}{\partial t}\big|_{t=0} \gamma_v(t) = \gamma_v'(0) = v$
- since $v \in T_pM$ was **arbitrary** the claim follows □

**Note:** Strictly speaking, we **identified** $T_0T_pM$ **with** $T_pM$ for the domain of $d\exp_p$ via the **canonical isomorphism** $(0, v) = v$.

**Remark**

If $\nabla$ is **geodesically complete**, exp is defined on $TM$. This however does **not** mean that there exists $p \in M$, such that $\exp_p$ is a **diffeomorphism**.

**Exercise**

- Show that for any $p \in \mathbb{R}^n$, $\exp_p$ defined on $T_p\mathbb{R}^n$ with respect to the canonical connection is a **diffeomorphism**.
- Show that if $M$ is **compact** and $\nabla$ is **any connection** in $TM \to M$, $\exp_p$ **is never a diffeomorphism for all** $p \in M$, independent of its domain $V_p \subset T_pM$.

Aside from geodesics, it is also common to study curves fulfilling a similar but weaker requirement:

**Definition**

A smooth curve $\gamma : I \to M$ is called **pregeodesic** with respect to a connection in $TM \to M$ if it has a reparametrisation as a geodesic, that is if there exists a diffeomorphism $f : I' \to I$, such that $\gamma \circ f$ is a geodesic.

Pregeodesics fulfil the following equation similar to the **geodesic equation**:

**Lemma**

Any given **pregeodesic** $\gamma : I \to M$ **with respect to a connection** $\nabla$ in $TM \to M$ fulfils $\nabla_{\gamma'}\gamma' = F\gamma'$ for some smooth function $F : I \to \mathbb{R}$.

**Proof:** Follows from writing out $\nabla_{(\gamma\circ f)'}(\gamma \circ f)'$ using our local formula for $\gamma \circ f$ a geodesic. □

We have not yet given a **geometric reason** as to **why** one would study geodesics in the first place. To do so we introduce the **energy functional** for curves with compact domain.

## Definition

Let $(M, g)$ be a **pseudo-Riemannian manifold** and $\gamma : [a, b] \to M$ be a **smooth curve**. The **energy functional** evaluated at $\gamma$, or simply **energy of** $\gamma$, is given by

$$E(\gamma) := \frac{1}{2} \int_a^b g(\gamma', \gamma') dt.$$

**Note:**

- Compare the above with the definition of the **length** $L(\gamma)$ of $\gamma$ for $(M, g)$ Riemannian!
- $E(\gamma)$ can also be defined for **piecewise smooth curves**, see discussion in lecture notes.
- We did **not** specify a structure on the domain of $E$. This is, in general, a very **difficult task**.

For our purposes we need to understand how to **perturb** a given curve $\gamma : [a, b] \to M$ in a "good" way, so that we can study the infinitesimal change in $E(\gamma)$.

## Definition

Let $\gamma : [a, b] \to M$ be a smooth curve and $\varepsilon > 0$. A smooth map $\eta : (-\varepsilon, \varepsilon) \times [a, b] \to M$ is called **variation of** $\gamma$ if $\eta(0, t) = \gamma(t)$ for all $t \in [a, b]$. $\eta$ is called **variation with fixed endpoints of** $\gamma$ if $\eta(s, a) = \gamma(a)$ and $\eta(s, b) = \gamma(b)$ for all $s \in (-\varepsilon, \varepsilon)$. The vector field $V$ along $\gamma$, $V_{\gamma(t)} = \frac{\partial \eta}{\partial s}(0, t) \in T_{\gamma(t)}M$, is the **variational vector field** corresponding to $\eta$.

**Note:** Geometrically, we understand $\eta$ as a **smooth family of curves** containing $\gamma$. Furthermore, independently of the chosen **variation with fixed endpoints**, $V_{\gamma(a)} = V_{\gamma(b)} = 0$.

**Question:** Can every vector field along a curve $\gamma$ with compact domain be **realised as the variational vector field** of a variation of $\gamma$?
**Answer:** Yes!

### Lemma

Let $\gamma : [a, b] \to M$ be a smooth curve and $V \in \Gamma_\gamma(TM)$. Then there exists a variation $\eta$ of $\gamma$, such that $V$ is the **variational vector field** of $\eta$. If $V_{\gamma(a)} = V_{\gamma(b)} = 0$, $\eta$ can be chosen to be variation with **fixed endpoints**.

**Proof:**

- fix a **Riemannian metric $g$ on $M$** with Levi-Civita connection $\nabla$
- let $\exp : V \to M$ denote the **corresponding exponential map**
- we now **define a variation of $\gamma$** via

$$\eta : (-\varepsilon, \varepsilon) \times [a, b] \to M, \quad \eta(s, t) := \exp(sV_{\gamma(t)})$$

for $\varepsilon > 0$ **small enough**

(continuation of proof)

- we can always find such an $\varepsilon$ by the **compactness of** $[a, b]$ and the **smoothness of** $V$
- if $V$ **vanishes at** $\gamma(a)$ **and** $\gamma(b)$, $\eta$ has the property $\eta(s, a) = \gamma(a)$ and $\eta(s, b) = \gamma(b)$ for all $s \in (-\varepsilon, \varepsilon)$
- we check with a calculation as the one in the proof of Proposition B
$$\frac{\partial \eta}{\partial s}(0, t) = V_{\gamma(t)}$$
for all $t \in [a, b]$
- hence, $\eta$ fulfils the required properties of this lemma □

We can now can describe the **infinitesimal change** in $E(\gamma)$ with respect to a variational vector field:

## Lemma

Let $(M, g)$ be a pseudo-Riemannian manifold with Levi-Civita connection $\nabla$. Then the first **variation of the energy** at a smooth curve $\gamma : [a, b] \to M$ **with respect to a given variational vector field** $V \in \Gamma_\gamma(TM)$ with a choice of corresponding variation of $\gamma$, $\eta : (-\varepsilon, \varepsilon) \times [a, b] \to M$, $\eta : (s, t) \mapsto \eta(s, t)$, is given by $\frac{\partial}{\partial s}\big|_{s=0} E(\eta(s, \cdot))$ and fulfils

$$\frac{\partial}{\partial s}\bigg|_{s=0} E(\eta(s, \cdot))$$

$$= -\int_a^b g(V, \nabla_{\gamma'}\gamma')dt + g(V_{\gamma(b)}, \gamma'(b)) - g(V_{\gamma(a)}, \gamma'(a)).$$

In the special case that $V$ **vanishes at the start- and end-point of** $\gamma$, we have have

$$\frac{\partial}{\partial s}\Big|_{s=0} E(\eta(s, \cdot)) = -\int_a^b g(V, \nabla_{\gamma'}\gamma')dt.$$

**Proof:**

- we will use the **Einstein summation convention**
- let $\eta' = \eta'(s,t) = \frac{\partial \eta}{\partial t}$ denote the **velocity vector field** of the family of smooth curves $\eta$ for $s$ fixed
- we calculate

$$\frac{\partial}{\partial s}\bigg|_{s=0} E(\eta(s,\cdot)) = \frac{1}{2}\int_a^b \frac{\partial}{\partial s}\bigg|_{s=0} g(\eta',\eta')dt$$

$$= \frac{1}{2}\int_a^b \nabla_V(g(\eta',\eta'))dt = \int_a^b g(\gamma',\nabla_V \eta')dt$$

- for the last equality we have used that $\nabla$ is **metric**
- for the next step we **need to prove that** $\nabla_V \eta' = \nabla_{\gamma'} V$
- $\rightsquigarrow$ use local coordinates, that is fix $p \in \gamma([a,b])$ and **choose local coordinates** $(x^1,\ldots,x^n)$ on an **open neighbourhood of** $p \in M$

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(continuation of proof)

- denote $V^s = V^s_{\eta(s,t)} := \frac{\partial \eta^k}{\partial s}\frac{\partial}{\partial x^k}$, so that $V^0 = V$
- $\rightsquigarrow$ **suffices** to show that $(\nabla_{V^s}\eta')|_{s=0} = \nabla_{\gamma'} V$
- using our **local formula** we find

$$\nabla_{V^s}\eta' = \left(\frac{\partial^2 \eta^k}{\partial s \partial t} + \frac{\partial \eta^i}{\partial s}\frac{\partial \eta^j}{\partial t}\Gamma^k_{ij}\right)\frac{\partial}{\partial x^k}$$

and

$$\nabla_{\gamma'} V = \left(\frac{\partial^2 \eta^k}{\partial t \partial s} + \frac{\partial \eta^i}{\partial t}\frac{\partial \eta^j}{\partial s}\Gamma^k_{ij}\right)\bigg|_{s=0}\frac{\partial}{\partial x^k}.$$

- $\nabla$ being **torsion-free** is **equivalent** to the **Christoffel symbols being symmetric in the lower indices**
- hence, the above local formulas for $(\nabla_{V^s}\eta')|_{s=0}$ and $\nabla_{\gamma'} V$ indeed **coincide**
- since $p \in \gamma([a,b])$ was **arbitrary** we deduce that the equality holds **for all** $t \in [a,b]$

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(continuation of proof)

- hence we obtain with partial integration

$$\int_a^b g(\gamma', \nabla_V \eta')dt$$

$$= \int_a^b g(\gamma', \nabla_{\gamma'} V)dt$$

$$= \int_a^b \left( \frac{\partial}{\partial t} g(\gamma', V) - g(\nabla_{\gamma'}\gamma', V) \right) dt$$

$$= g(V_{\gamma(b)}, \gamma'(b)) - g(V_{\gamma(a)}, \gamma'(a)) - \int_a^b g(V, \nabla_{\gamma'}\gamma')dt$$

- **reordering** the above equation finishes the proof □

### Corollary

Geodesics defined on a compact interval with respect to the Levi-Civita connection of a pseudo-Riemannian are **critical points** of the energy functional in the sense that the **first variation of the energy** with respect to variations with fixed end points **vanishes**.

The converse also holds true:

### Exercise

A curve in a pseudo-Riemannian manifold defined on a compact interval is a **geodesic** with respect to the Levi-Civita connection **if it is a critical point** of the energy functional.

Furthermore, you should try to solve:

### Exercise A

Find a formula for the **first variation of the length** of a curve in a **Riemannian manifold**. Are geodesics also **critical points of the length functional** in our sense?

## Remark

In **Riemannian geometry**, one can show that geodesics with respect to the Levi-Civita connection and with compact domain are **not just critical points** of the energy and length functional, but also **local minimisers**. This means that for every variation with fixed endpoints $\eta : (-\varepsilon, \varepsilon) \times [a, b] \to M$ of a geodesic $\gamma : [a, b] \to M$ in $(M, g)$, $E(\eta(s, \cdot)) \geq E(\gamma)$ for $\varepsilon$ small enough.

**References:**


- J.M. Lee, *Riemannian Manifolds – An Introduction to Curvature*, Springer GTM 176 (1997)
- C. Bär, *Differential Geometry*, lecture notes (2013)
- O. Goertsches, *Differentialgeometrie*, lecture notes (2014) (in German)

The exponential map at a fixed point can be used to construct **coordinates** on a given manifold with **particularly nice properties** if $\exp_p$ comes from the **Levi-Civita connection**.

## Definition

Let $M$ be a smooth manifold with connection $\nabla$ in its tangent bundle. Suppose that $V \subset T_pM$ is a **star-shaped open neighbourhood of the origin**, such that $\exp_p : V \to \exp_p(V)$ is a **diffeomorphism**. Then $U = \exp_p(V)$ is an open neighbourhood of $p \in M$ and is called **normal neighbourhood** of $p \in M$. Let $U \subset M$ be such a normal neighbourhood of a **fixed** $p \in M$. Then the exponential map at $p$ can be used to **define local coordinates** $(x^1, \ldots, x^n)$ near $p$ as follows: Choose a **basis** $\{v_1, \ldots, v_n\}$ of $T_pM$ and define coordinates **implicitly** via

$$\exp_p\left(\sum_{i=1}^{n} x^i(q) v_i\right) = q$$

for all $q \in U$. This just means that the $x^i$ are the **prefactor functions** of $\exp_p^{-1}$ **written in the basis** $\{v_1, \ldots, v_n\}$.
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## Definition (continuation)

**Smoothness** of the $x^i$ follows from the **implicit function theorem**. If $(M, g)$ is a **pseudo-Riemannian manifold** with **Levi-Civita connection** $\nabla$, normal coordinates at $p \in M$ with respect to an **orthonormal basis** $\{v_1, \ldots, v_n\}$ of $T_pM$ are called **Riemannian normal coordinates** at $p \in M$. If $(M, g)$ is **Riemannian** and $V = B_r(0) = \{v \in T_pM \mid g_p(v, v) < r\}$ for some $r > 0$, the corresponding domain of the Riemannian normal coordinates $B_r^g(p) := \exp_p(B_\varepsilon(0))$ is called **geodesic ball of radius** $r$ **centred at** $p$ in $M$. The **upper index** $g$ indicates the corresponding Riemannian metric.

One application of Riemannian normal coordinates is:

## Lemma

Any two points of a connected pseudo-Riemannian manifold $(M, g)$ can be **connected by a piecewise smooth curve**, such that **every smooth segment of that curve is a geodesic**.

**Proof:** Exercise!

Apart from connecting points with piecewise geodesics, Riemannian normal coordinates have the property that certain **geometric quantities** are, at the reference point, of a **very simple form**.

## Proposition C

Let $(M, g)$ be a **pseudo-Riemannian manifold** with Levi-Civita connection $\nabla$ and let $\varphi = (x^1, \ldots, x^n)$ be **Riemannian normal coordinates near** $p \in M$ corresponding to a choice of orthonormal basis $\{v_1, \ldots, v_n\}$ of $T_pM$. Then the **prefactors of** $g$ **written locally** as $\sum g_{ij} dx^i dx^j$ fulfil

$$g_{ij}(p) = \varepsilon_{ij}$$

for all $1 \leq i, j \leq n$, where $\varepsilon_{ij} = g(v_i, v_j)$. The **Christoffel symbols** of $\nabla$ and **all partial derivatives** of the local smooth functions $g_{ij}$ **vanish** at $p$, that is

$$\Gamma^k_{ij}(p) = 0, \quad \frac{\partial g_{ij}}{\partial x^k}(p) = 0$$

for all $1 \leq i, j, k \leq n$. (continued on next page)

## Proposition C (continuation)

If $\gamma_w : (-\varepsilon, \varepsilon) \to M$, $\gamma_w(0) = p$, $\gamma_w'(0) = w \in T_pM$, is a **geodesic starting at** $p \in M$ such that its image is contained in the domain of $\varphi$, $\varphi \circ \gamma_w$ **is of the form**

$$\varphi(\gamma_w(t)) = tw$$

for all $t \in (-\varepsilon, \varepsilon)$.

**Proof:** (next page)

(continuation of proof)

- for $g_{ij}(p) = \varepsilon_{ij}$ we show that $v_k = \frac{\partial}{\partial x^k}\Big|_p$ for all $1 \leq k \leq n$
- $x^k(p) = 0$ for all $1 \leq k \leq n$ **by construction** implies (after, as before, identifying $T_0 T_p M \cong T_p M$)

$$\sum_{k=1}^{n} d \exp_p\Big|_0 (v_k) \otimes dx^k|_p = \mathrm{id}_{T_p M}. \qquad (1)$$

- **on the other hand** we know by Proposition B that $d \exp_p |_0(v_k) = v_k$ for all $1 \leq k \leq n$
- applying both sides of (2) to $\frac{\partial}{\partial x^k}\Big|_p$ **proves our claim** and, hence, $g_{ij}(p) = \varepsilon_{ij}$
- next, note that $\frac{\partial g_{ij}}{\partial x^k}(p) = 0$ implies with the help of **explicit local formula** for Christoffel symbols that all Christoffel symbols at $p$ **must also vanish**
- In order to prove $\frac{\partial g_{ij}}{\partial x^k}(p) = 0$ we first show that the local form of geodesics $\varphi \circ \gamma_w$ is of the **claimed form**

(continued on next page)

(continuation of proof)

- by **construction** of the exponential map have $\gamma_w(t) = \exp_p(tw)$ for all $t \in (-\varepsilon, \varepsilon)$
- writing $w = \sum_{k=1}^{n} w^k v_k$, we have **by definition of Riemannian normal coordinates**

$$\gamma_w(t) = \exp_p(tw)$$
$$= \exp_p\left(\sum_{k=1}^{n} tw^k v_k\right) = \exp_p\left(\sum_{k=1}^{n} x^k(\gamma_w(t))v_k\right),$$

  showing that $\varphi(\gamma_w(t)) = tw$ for all $t \in (-\varepsilon, \varepsilon)$ **as claimed**
- writing down the **geodesic equation** for $\gamma_w$ in our **local coordinates** at $p$ with $\ddot{x}^k(0) = \frac{\partial^2(x^k(\gamma_w))}{\partial t^2}(0) = 0$ and $\dot{x}^k(0) = w^k$ for all $1 \leq k \leq n$ shows that

$$\sum_{i,j=1}^{n} \Gamma_{ij}^k(p) w^i w^j = 0 \quad \forall 1 \leq k \leq n$$

(continued on next page)

(continuation of proof)

- this holds for **arbitrary initial condition** for the geodesic $\gamma_w'(0) = w \in T_pM$, this proves that **for each fixed** $1 \leq k \leq n$, $(\Gamma_{ij}^k(p))_{ij}$ **viewed as symmetric bilinear form** on $T_pM \times T_pM$ must **identically vanish**
- hence, $\Gamma_{ij}^k(p) = 0$ **for all** $1 \leq i, j, k \leq n$
- for the **vanishing of the partial derivatives** of each $g_{ij}$ **at** $p$, observe that $\nabla$ being **metric** implies

$$\frac{\partial g_{ij}}{\partial x^k} = \frac{\partial}{\partial x^k}\left(g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\right)$$

$$= g\left(\sum_{\ell=1}^{n} \Gamma_{ki}^{\ell} \frac{\partial}{\partial \ell}, \frac{\partial}{\partial j}\right) + g\left(\sum_{\ell=1}^{n} \Gamma_{kj}^{\ell} \frac{\partial}{\partial \ell}, \frac{\partial}{\partial i}\right)$$

  for all $1 \leq k \leq n$
- **evaluating** the above equation **at** $p$ and using that **all Christoffel symbols vanish at** $p$ yields the desired result □

**Warning**

In Proposition C we have seen that with the **right choice of coordinates**, any pseudo-Riemannian metric and Levi-Civita connection can be brought to a very simple form **at a chosen point**. While this **works** of course **for every point** in the manifold, this does **not** mean that every pseudo-Riemannian metric is **locally** of the form $g_{ij} = \varepsilon_{ij}$ on some open neighbourhood of our chosen reference point, **this can in general only be achieved at said point!** Otherwise, every manifold would be **flat**, and comparing with the upcoming lecture about **curvature** shows that this is clearly **not the case**.

Proposition C implies that we have some **control** over the **Taylor expansion** of a pseudo-Riemannian metric at a chosen point.

## Corollary

The **Taylor expansion** of $g_{ij}$ **in Riemannian normal coordinates** $(x^1, \dots, x^n)$ at their reference point $p \in M$ is of the form

$$g_{ij} = \varepsilon_{ij} + O\left(\sum_{k=1}^{n} (x^k)^2\right).$$

Lastly, we will discuss some **advanced results** in this setting that we will not prove, but you should still have heard about.

## Remark

- We have seen that geodesics with **compact domain** in Riemannian manifolds are **critical points of the energy functional**, and solving Exercise A shows that they are in fact also **critical points of the length functional**. One can, however, show more and prove that they are not just any type of critical point but **local minimisers**, meaning that for any variation $\eta$ of $\gamma$ with fixed endpoints, $E(\eta(s,\cdot)) \geq E(\gamma)$ and $L(\eta(s,\cdot)) \geq L(\gamma)$ for $s$ small enough. **Reference:** Chapter 6 in J.M. Lee, *Riemannian Manifolds – An Introduction to Curvature*, Springer GTM 176 (1997)

(continued on next page)

## Remark (continuation)

- An other way to study Riemannian manifolds is in the context of **metric geometry**. In fact, every Riemannian metric $g$ on a smooth manifold $M$ induces the structure of a **metric space** on $M$, which in turn induces a topology on $M$. It turns out that the induced topology on $M$ coincides, **independently** of the Riemannian metric $g$, with the **initial topology** on $M$.
- We have interpreted variations of curves as a **family of curves** depending on one parameter. In the case that $\gamma : I \to M$ is a geodesic and $\eta : (-\varepsilon, \varepsilon) \times I \to M$ is a variation of $\gamma$, are there **choices** for $\eta$, such that **every** $\eta(s, \cdot) : I \to M$ **is a geodesic**, not just $\eta(0, \cdot) = \gamma$? The answer is **yes**, and the **corresponding variational vector fields** are called **Jacobi fields**.

An extremely useful theorem in global Riemannian geometry is the following:

**Theorem (Hopf-Rinow)**

Let $(M, g)$ be a **Riemannian manifold**. Then the following are **equivalent**:

- $(M, g)$ is **geodesically complete**.
- $M$ with the induced metric[a] from the Riemannian metric $g$ is **complete as a metric space**.
- Every **closed and bounded**[b] **subset** of $M$ is **compact**.

[a] As in metric space.
[b] W.r.t. the induced metric.

**Proof:** See Ch. 5, Thm. 21 in B. O'Neill, *Semi-Riemannian Geometry With Applications to Relativity* (1983), Pure and Applied Mathematics, Vol. **103**, Academic Press, NY.

The Hopf-Rinow Theorem can be used to prove the following lemmata about **geodesic completeness**. Note that they are **in practice** actually useful.

**Lemma**

A Riemannian manifold $(M, g)$ is geodesically complete **if and only if** every curve with image not contained in any compact set has **infinite length**.

**Proof:**

- if $(M, g)$ is geodesically complete, a curve $\gamma$ that is **not contained in any compact set** is by the Hopf-Rinow Theorem in particular **not contained in the closure of the geodesic ball** $B_r^g(\gamma(t_0))$ for any $t_0$ in the domain of $\gamma$ and any $r > 0$
- hence, $\gamma$ has **infinite length**

(continued on next page)

(continuation of proof)

- if $(M, g)$ is **geodesically incomplete**, we can find an **inextensible geodesic** $\gamma : [0, a) \to M$, $a > 0$, of **unit speed** **[**note: $L(\gamma) = a < \infty$**]**
- suppose that $\gamma([0, a))$ is **contained in a compactum** $K \subset M$
- then $\gamma$ **converges in** $K$ and can thus be **extended as a geodesic**, which is a **contradiction** □

## Lemma

- Let $M$ be a smooth manifold and $g, h$ **Riemannian metrics** on $M$. Assume that for all $p \in M$ and all $v \in T_pM$, $h_p(v, v) \geq g_p(v, v)$, or $h \geq g$ for short. If $(M, g)$ is **geodesically complete**, $(M, h)$ is **also geodesically complete**.
- Let $(M, g)$ be a **Riemannian manifold**. If there **exists** $R > 0$, such that $\overline{B^g_R(p)}$ is **compactly embedded** in $M$ **for all** $p \in M$, then $(M, g)$ is **geodesically complete**.

**Proof:** Exercise!

# END OF LECTURE 17

**Next lecture:**

- curvature