Differential geometry
Lecture 16: Parallel transport and the Levi-Civita connection

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Recap of lecture 15:

- introduced **connections in vector bundles**
- explained **local form** of $\nabla_X s$ using **connection 1-forms**
- defined **Christoffel symbols** for connections in tangent bundle
- studied their **transformation behaviour**
- obtained connection in $T^{r,s}M \to M$ from connection in $TM \to M$ via **tensor derivation property**
- **erratum:** messed up the indices of the Christoffel symbols of the example connection in $T\mathbb{R}^2 \to \mathbb{R}^2$ in polar coordinates, correct would have been

$$\Gamma^r_{\varphi\varphi} = -r, \quad \Gamma^\varphi_{r\varphi} = \Gamma^\varphi_{\varphi r} = \frac{1}{r}, \quad 0 \text{ else}$$

Connections give meaning to the term "constant" for sections of the respective vector bundle. Next, we will study tensor fields that are only defined **along curves**.

Definition

An (r,s)-**tensor field** $A = A_\gamma$ **along a curve** $\gamma : I \to M$ is a smooth map

$$A_\gamma : I \to T^{r,s}M, \quad t \mapsto A_{\gamma(t)} \in T^{r,s}_{\gamma(t)}M.$$

If γ is an embedding and thus $\gamma(I)$ is a submanifold of M, an (r,s)-tensor field A_γ along γ is simply a **parametrisation** of a smooth section in $T^{r,s}M|_{\gamma(I)} \to \gamma(I)$.

Note: The above definition **extends** what is allowed as vector field along curves. Recall that until this point, we understood under this term the pushforward of a vector field on an interval $I \subset \mathbb{R}$ to a smooth manifold M via a smooth curve $\gamma : I \to M$.

In general, (r,s)-tensor fields along curves can not be extended to (r,s)-tensor fields on the ambient manifold.
Question: Can we always find local extensions near $\gamma(t_0)$?
Answer: Yes if $\gamma'(t_0) \neq 0$!

Lemma

Let $\gamma : I \to M$ be a smooth curve and suppose that $\gamma'(t_0) \neq 0$. Let further A_γ be an (r,s)-tensor field along γ. Then there exists an open interval $I' \subset I$, $t_0 \in I'$, such that $A_\gamma|_{I'}$ is the **restriction of an** (r,s)**-tensor field** $\overline{A} \in \mathcal{T}^{r,s}(M)$.

Proof:

- use ansatz similar to local rectification of vector fields
- after restricting to I' and using a suitable choice of local coordinates $\varphi = (x^1, \dots, x^n)$ on $U \subset M$ with $\gamma(I') \subset U$, γ is of the form $t \mapsto \varphi^{-1}(t, 0, \dots, 0)$, so that $x^1(\gamma(t)) = t$ and $x^i(\gamma(t)) = 0$ for $2 \leq i \leq n$.

(continued on next page)

(continuation of proof)

- hence, if

$$A_{\gamma(t)} = \sum f^{i_1 \dots i_r}{}_{j_1 \dots j_s}(t) \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_s},$$

  with all $f^{i_1 \cdots i_r}{}_{j_1 \dots j_s} : I' \to \mathbb{R}$ smooth, the tensor field

$$\overline{A} = \sum \left(f^{i_1 \dots i_r}{}_{j_1 \dots j_s} \circ x^1 \right) \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_s}$$

  fulfils the requirements of this lemma □

Remark: One way the statement of the above lemma can go wrong for $\gamma'(t_0) = 0$ is when γ is the **constant curve** but A_γ is **not constant**.

Question: How do we measure the **infinitesimal change** of a tensor field along a curve using a connection in the corresponding tensor bundle of the ambient manifold?
Answer:

Proposition

Let M be a smooth manifold and ∇ a connection in $TM \to M$. Let $\gamma : I \to M$ be a smooth curve and denote the set of vector fields along γ by $\Gamma_\gamma(TM)$. Then there exists a **unique** $\mathbb{R}$**-linear map**

$$\tfrac{\nabla}{dt} : \Gamma_\gamma(TM) \to \Gamma_\gamma(TM),$$

such that

$$\tfrac{\nabla}{dt}(fX) = \frac{\partial f}{\partial t} X + f \tfrac{\nabla}{dt} X$$

for all $f \in C^\infty(I)$ and all $X \in \Gamma_\gamma(TM)$ and, if $X = \overline{X}|_{\gamma(I)}$,

$$\tfrac{\nabla}{dt} X = \nabla_{\gamma'} \overline{X}$$

for all $t \in I$.

Proof: (next page)

- **suppose** that such a map $\frac{\nabla}{dt}$ exists $\rightsquigarrow$ show that it is then **unique**
- if $\gamma'(t_0) = 0$, set $\frac{\nabla}{dt} X|_{t=t_0} = 0 \; \forall X \in \Gamma_\gamma(TM)$
- this is **compatible with the tensoriality** in the first argument of any connection, so that $\left(\frac{\nabla}{dt} X\right)\Big|_{t=t_0} = 0$ for all vector fields X along γ that are restrictions $X = \overline{X}_\gamma$ of vector fields $\overline{X} \in \mathfrak{X}(M)$
- if $\gamma'(t_0) \neq 0$, choose local coordinates as in **extension lemma** $\varphi = (x^1, \ldots, x^n)$ with $\gamma = \varphi^{-1}(t, 0, \ldots, 0)$ near t_0, find local formula

$$\left(\frac{\nabla}{dt} X\right)\Big|_{t=t_0} = \left(\nabla_{\gamma'} \overline{X}\right)\Big|_{t=t_0}$$
$$= \sum_{k=1}^{n} \left(\frac{\partial X^k}{\partial t}(t_0) + \sum_{i,j=1}^{n} \frac{\partial \gamma^i}{\partial t}(t_0) X^j(t_0) \Gamma^k_{ij}(t_0) \right) \frac{\partial}{\partial x^k}\Big|_{t=t_0}$$

for all $X = \sum_{k=0}^{n} X^k(t) \frac{\partial}{\partial x^k}$ with local extension

$\overline{X} = \sum_{k=1}^{n} \left(X^k \circ x^1\right) \frac{\partial}{\partial x^k}$ (continued on next page)

(continuation of proof)

- hence: if the operator $\frac{\nabla}{dt}$ exists, it is **uniquely determined by the connection** ∇
- on the other hand observe that the local formula of $\left(\frac{\nabla}{dt}X\right)\Big|_{t=t_0}$ for $\gamma'(t_0) \neq 0$ **defines** by the locality property of connections an operator $\frac{\nabla}{dt}$ fulfilling the requirements of this proposition, at least in fixed chosen local coordinates
- to check that the operator extends to $\Gamma_\gamma(TM)$, one needs to check that it **transforms as a connection** and is thus **independent of the chosen local extension** $\overline{X}$ of X [exercise!] □

Definition

The linear differential operator $\nabla_{\gamma'}$ is called **covariant derivative along** γ. It has the **local form**

$$\nabla_{\gamma'} X = \sum_{k=1}^{n} \left(\frac{\partial X^k}{\partial t} + \sum_{i,j=1}^{n} \frac{\partial \gamma^i}{\partial t} X^j \Gamma^k_{ij} \right) \frac{\partial}{\partial x^k}$$

for all $X \in \Gamma_\gamma(TM)$ locally given by $X = \sum_{k=1}^{n} X^k \frac{\partial}{\partial x^k}$, $X^k = X^k(t) \in C^\infty(I) \ \forall\, 1 \leq i \leq n$.

Remark: More generally, one can define the covariant derivative along curves for **any** vector bundle $E \to M$ with a connection. From the tensor derivative property of connections one obtains:

Corollary

Let A be an (r,s)-tensor field on a smooth manifold M along $\gamma : I \to M$ with $r, s \geq 1$. Let $C : \mathcal{T}^{r,s}(M) \to \mathcal{T}^{r-1,s-1}(M)$ be any **contraction** (note: C **canonically extends** to (r,s)-tensor fields along curves). Then $C(\nabla_{\gamma'} A) = \nabla_{\gamma'}(C(A))$.

Notation: We will use $\nabla_{\gamma'}$ for $\frac{\nabla}{dt}$, this is up to preference.

Having defined what covariant differentiation along a curve is allows us to define what it means for a vector field along a curve to be **parallel** along said curve:

Definition

Let $X \in \Gamma_\gamma(TM)$ be a vector field along a smooth curve $\gamma : I \to M$ and let ∇ be a connection in $TM \to M$. X is called **parallel along** γ, or simply **parallel**, if $\nabla_{\gamma'} X = 0$.

Remark: One similarly defines **parallel tensor fields** along curves. E.g. for 1-forms along curves $\omega \in \Gamma_\gamma(T^*M)$ we find that $\nabla_{\gamma'}\omega = 0$ if and only if

$$\frac{\partial(\omega(X))}{\partial t} - \omega(\nabla_{\gamma'} X) = 0$$

for all $X \in \Gamma_\gamma(TM)$.

Our notion of "parallel along curves" allows us to **translate vectors** (or covectors, tensor powers of vectors and covectors) **in a parallel way** along curves.

Theorem

Let ∇ be a connection in $TM \to M$, $\gamma : I \to M$, $t_0 \in I$, be a smooth curve with non-vanishing velocity, and $v \in T_{\gamma(t_0)}M$. Then there exists a **unique vector field along** γ, $X \in \Gamma_\gamma(TM)$, such that X is parallel along γ and $X_{\gamma(t_0)} = v$. This means that X is the **unique solution to the initial value problem**

$$\nabla_{\gamma'} X = 0, \quad X_{\gamma(t_0)} = v.$$

Proof:

- locally, $\nabla_{\gamma'} X = 0$ is an ODE, hence has **locally** unique solutions
- for global statement, need to deal with cases where $\gamma(I)$ is **not covered by a single chart**
- Exercise! Alternatively see Thm. 4.11 in Lee's "Riemannian Manifolds. An Introduction to Curvature". □

Note: The latter theorem can be formulated not just for vector fields, but sections in any vector bundle with connection, e.g. $T^{r,s}M \to M$.

Example

Let $\gamma : \mathbb{R} \to \mathbb{R}^2$, $\gamma : t \mapsto \binom{t}{1}$ and

$$X = X_\gamma = (\gamma(t), \binom{1}{1}) \in \Gamma_\gamma(T\mathbb{R}^2).$$

Let ∇ be a connection in $T\mathbb{R}^2 \to \mathbb{R}^2$ defined by setting its **Christoffel symbols in canonical coordinates** all equal to 0. Then X is **parallel along** γ, i.e. $\nabla_{\gamma'} X = 0$, meaning that X solves the initial value problem of parallelly transporting $v = (\gamma(0), \binom{1}{1})$ along γ.

Question: Can we use covariant derivatives along curves to **recover** their defining connection?
Answer: Yes! $\rightsquigarrow$ Need the following property of parallel translations:

Lemma

Let ∇ be a connection in $TM \to M$ and $\gamma : I \to M$ a smooth curve. Consider parallel translations along γ as **maps**

$$P_{t_0}^{t}(\gamma) : T_{\gamma(t_0)}M \to T_{\gamma(t)}M,$$

mapping initial values $v \in T_{\gamma(t_0)}M$, $t_0 \in I$, of the differential equation $\nabla_{\gamma'}X = 0$, to the value of its uniquely solution X at $t \in I$, namely $X_{\gamma(t)} \in T_{\gamma(t)}M$. Then $P_{t_0}^{t}(\gamma)$ is a **linear isomorphism** for all $t_0, t \in I$.

Proof:

- linearity of $P_{t_0}^{t}(\gamma)$ follows by observing that whenever X solves $\nabla_{\gamma'}X = 0$ for initial value $v \in T_{\gamma t_0}M$, cX is **also parallel** along γ and is the unique solution of the parallel transport equation for initial value $cv \in T_{\gamma(t_0)}M$

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(continuation of proof)

- to see that $P_{t_0}^t(\gamma)$ is invertible, **fix** $t \in I$ and let $\widetilde{\gamma}(s) := \gamma(t-s)$
- $\rightsquigarrow$ the **parallel transport with respect to** $\widetilde{\gamma}$ from $s=0$ to $s = t - t_0$, $P_0^{t-t_0}(\widetilde{\gamma}) : T_{\gamma(t)}M \to T_{\gamma(t_0)}M$ is precisely the **inverse of** $P_{t_0}^t(\gamma) : T_{\gamma(t_0)}M \to T_{\gamma(t)}M$, which follows from $\nabla_{\widetilde{\gamma}'}(X \circ (t-s)) = 0$ for X being the unique solution of $\nabla_{\gamma'}X = 0$ with fixed initial value in $T_{\gamma(t_0)}M$ □

With this result we can describe a connection completely by its covariant derivatives along curves:

Proposition A

Let ∇ be a connection in $TM \to M$ and $X, Y \in \mathfrak{X}(M)$. For $p \in M$ arbitrary let $\gamma : (-\varepsilon, \varepsilon) \to M$, $\varepsilon > 0$, be an **integral curve** of X with $\gamma(0) = p$, and let $P_{t_0}^t$ denote the corresponding parallel transport maps. Then

$$(\nabla_X Y)_p = \frac{\partial}{\partial t}\bigg|_{t=0} P_t^0(\gamma) Y_{\gamma(t)}.$$

Proof: (next page)

(continuation of proof)

- note: $t \mapsto P_t^0(\gamma) Y_{\gamma(t)}$ is **smooth**, follows from the smoothness of the local prefactors of the defining differential equation in local coordinates
- the smooth manifold structure in $T_p M$ is given by its **linear isomorphy** to $\mathbb{R}^n$
- also note: $P_t^0(\gamma) Y_{\gamma(t)} \in T_p M$ for all $t \in (-\varepsilon, \varepsilon)$, so it **makes sense** to take its time derivative
- choose **basis** $\{v_1, \dots, v_n\}$ of $T_p M$, e.g. via local charts
- $\rightsquigarrow$ for all $1 \leq i \leq n$, $V_i = V_i|_{\gamma(t)} := P_0^t(\gamma) v_i$ defines a **parallel vector field** along γ, i.e. $\nabla_{\gamma'} V_i = 0$
- hence $\{V_1, \dots, V_n\}$ is a **parallel frame** of TM **along** $\gamma|_{(-\varepsilon,\varepsilon)}$, meaning that each vector field along γ that is the restriction of a vector field on the ambient manifold can be written as a $C^\infty((-\varepsilon, \varepsilon))$-linear combination of its elements

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(continuation of proof)

- thus we can write

$$Y_\gamma = \sum_{i=1}^{n} f^i V_i,$$

$f^i \in C^\infty((-\varepsilon, \varepsilon))$ for all $1 \leq i \leq n$

- using $(\nabla_X Y)_p = \nabla_{\gamma'} Y_\gamma|_{t=0}$ we calculate

$$\nabla_{\gamma'} Y_\gamma|_{t=0} = \sum_{i=1}^{n} \left(\frac{\partial f^i}{\partial t} V_i + f^i \nabla_{\gamma'} V_i \right) \Big|_{t=0} = \sum_{i=1}^{n} \frac{\partial f^i}{\partial t}(0) v_i$$

- On the other hand, we have for all $t \in (-\varepsilon, \varepsilon)$

$$P_t^0(\gamma) Y_{\gamma(t)} = P_t^0(\gamma) \left(\sum_{i=1}^{n} f^i(t) V_i|_{\gamma(t)} \right) = f^i(t) v_i, \qquad (1)$$

where we used that $P_t^0(\gamma) = \left(P_0^t(\gamma)\right)^{-1}$ and that, by construction, V_i is precisely the **parallel extension of** v_i **along** γ for all $1 \leq i \leq n$ (continued on next page)

(continuation of proof)

- $\rightsquigarrow$ taking the t-derivative at $t = 0$ of the right hand side of (1) finishes the proof □

The previous proposition has the following at first sight **surprising** consequence:

Corollary

Let ∇ be a connection in $TM \to M$ and $X, Y \in \mathfrak{X}(M)$. Then $(\nabla_X Y)_p$ depends only on X_p, **any** choice of smooth curve $\gamma : (-\varepsilon, \varepsilon) \to M$, $\varepsilon > 0$, with $\gamma'(0) = X_p$, and Y_γ, that is Y along γ.

The next definition will allow us to understand the "space" of connections in the tangent bundle better.

Definition

Let M be a smooth manifold and let ∇^1, ∇^2 be connections in $TM \to M$. Then the **difference tensor** $A \in \mathcal{T}^{1,2}(M)$ of ∇^1 and ∇^2 is defined via

$$A(X, Y) := \nabla^1_X Y - \nabla^2_X Y \quad \forall X, Y \in \mathfrak{X}(M).$$

After showing that the difference tensor is, **in fact**, a tensor field, this means that we can interpret the space of connections in $TM \to M$ as an **affine space** with basepoint any fixed connection ∇ and linear space $\mathcal{T}^{1,2}(M)$ with origin ∇.

Apart from its interpretation as the proper generalisation of derivatives for sections in vector bundles, we do not yet have a purely geometric interpretation of connections. Using the next definition will allow us to find such a property.

Definition

The **torsion tensor** $T \in \mathcal{T}^{1,2}(M)$ of a connection ∇ in $TM \to M$ is given by

$$T(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y]$$

for all $X, Y \in \mathfrak{X}(M)$. The connection ∇ is called **torsion-free** if $T \equiv 0$.

Remark: ∇ in $TM \to M$ is torsion free **if and only if** $\Gamma_{ij}^k = \Gamma_{ji}^k$ for all Christoffel symbols.

Remark

Consider for $n \geq 2$ the connection in $T\mathbb{R}^n \to \mathbb{R}^n$ with **vanishing** Christoffel symbols, **fix** $p \in \mathbb{R}^n$, and **choose two linearly independent vectors** $v, w \in T_p\mathbb{R}^n$. Let further $\varepsilon > 0$ and

$$\gamma_v := t \mapsto p + tv, \quad \gamma_w := t \mapsto p + tw.$$

For any $t > 0$, the four vectors

$$v, w, P_0^1(\gamma_v)w, P_0^1(\gamma_w)v$$

can be interpreted as the **edges of a parallelogram**. What is the proper **analogue for this picture** for general smooth manifolds M and connections in $TM \to M$? The answer lies in making $t > 0$ **infinitesimally small** and using Proposition A. We fix $p \in M$ and local coordinates $\varphi = (x^1, \dots, x^n)$ on $U \subset M$, $p \in U$.
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Remark (continuation)

For $1 \leq k \leq n$ and $\varepsilon > 0$ small enough, consider the smooth curves

$$\gamma_k : (-\varepsilon, \varepsilon) \to M, \quad x^\ell(\gamma_k(t)) = \delta^\ell_k t \quad \forall 1 \leq k, \ell \leq n,$$

so that $\gamma_k' = \frac{\partial}{\partial x^k}$. For any $i \neq j$, we obtain using Proposition A

$$\begin{aligned} & T\left(\tfrac{\partial}{\partial x^i}, \tfrac{\partial}{\partial x^j}\right)\Big|_p \\ &= \left(\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} - \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i}\right)\Bigg|_p \\ &= \frac{\partial}{\partial t}\bigg|_{t=0} \left(P^0_t(\gamma_i) \tfrac{\partial}{\partial x^j} - P^0_t(\gamma_j) \tfrac{\partial}{\partial x^i}\right) \\ &= \lim_{\substack{t \to 0 \\ t > 0}} \frac{\frac{\partial}{\partial x^i}\big|_p + P^0_t(\gamma_i) \frac{\partial}{\partial x^j} - \frac{\partial}{\partial x^j}\big|_p - P^0_t(\gamma_j) \frac{\partial}{\partial x^i}}{t}. \end{aligned}$$

Hence, the **“infinitesimal” parallelograms** spanned by any two different coordinate vectors and their parallel translations **close**, meaning that there is no “gap” when gluing the “infinitesimal” edges together.

Question: How do we combine the properties of a **pseudo-Riemannian metric** and a connection?
Answer: Define **metric** connections:

Definition

Let (M, g) be a pseudo-Riemannian manifold. A connection ∇ in $TM \to M$ is called **metric** if $\nabla g = 0$, that is

$$X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \tag{2}$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

How can we interpret the above definition? The answer is as follows.

Proposition

A connection in $TM \to M$ on a pseudo-Riemannian manifold (M, g) is metric **if and only if** its parallel transport maps $P_{t_0}^t(\gamma) : T_{\gamma(t_0)}M \to T_{\gamma(t)}M$ are **linear isometries** for all smooth curves $\gamma : I \to M$.

Proof: (next page)

(continuation of proof)

- all possible $P_{t_0}^t(\gamma)$ are linear isometries **if and only if** for all such $P_{t_0}^t(\gamma)$ and all $v, w \in T_pM$, $p = \gamma(t_0)$, the map

$$t \mapsto g_{\gamma(t)}(P_{t_0}^t(\gamma)v, P_{t_0}^t(\gamma)w)$$

  is constant
- by considering **affine reparametrisations** of curves by $t \to t + c$ for constant c, find that this holds **if and only if**

$$\left.\frac{\partial}{\partial t}\right|_{t=t_0} g_{\gamma(t)}(P_{t_0}^t(\gamma)v, P_{t_0}^t(\gamma)w) = 0$$

  for all parallel translations $P_{t_0}^t(\gamma)$
- viewing $P_{t_0}^t(\gamma)v$ and $P_{t_0}^t(\gamma)w$ as vector fields along γ, it now follows from the **tensor derivation property** of any connection ∇ that **if ∇ is metric**, the left hand side of the above equation always **vanishes**
- if one has problems seeing that, formally replace $\left.\frac{\partial}{\partial t}\right|_{t=0}$ by $\nabla_{\gamma'}|_{t=0}$

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(continuation of proof)

- for the other direction, suppose that $\frac{\partial}{\partial t}\big|_{t=t_0} g_{\gamma(t)}(P_{t_0}^t(\gamma)v, P_{t_0}^t(\gamma)w) = 0$ holds for all parallel translations
- let $X, Y, Z \in \mathfrak{X}(M)$, **fix** $p \in M$, construct a **local parallel frame** of TM along a curve γ fulfilling $\gamma'(0) = X_p$
- write Y_γ and Z_γ in that parallel frame and, using these local forms, check that indeed

$$X_p(g(Y, Z)) = \nabla_{\gamma'}(g_\gamma(Y_\gamma, Z_\gamma)|_{t=0}$$

  using the **tensor derivation property** of ∇ and Proposition A
- by $\frac{\partial}{\partial t}\big|_{t=t_0} g_{\gamma(t)}(P_{t_0}^t(\gamma)v, P_{t_0}^t(\gamma)w) = 0$ it then follows that ∇ is metric
- since X, Y, Z and p were **arbitrary**, it follows that ∇ **is indeed a metric connection** $\square$

While torsion-freeness and metric property alone do not determine a connection uniquely, the situation changes if both are assumed.

Definition

Let (M, g) be a pseudo-Riemannian manifold. A connection ∇ in $TM \to M$ is called **Levi-Civita connection** if it is **metric** and **torsion-free**.

Proposition

Let (M, g) be a pseudo-Riemannian manifold. Then there exists a **unique Levi-Civita connection** in $TM \to M$.

Proof: For the proof of this proposition we will introduce the so-called **Koszul formula**.

Proposition

Let (M, g) be a pseudo-Riemannian manifold and ∇ a connection in $TM \to M$. Then ∇ is the Levi-Civita connection of (M, g) if and only if it satisfies the **Koszul formula**

$$\begin{aligned} 2g(\nabla_X Y, Z) = &X(g(Y,Z)) + Y(g(X,Z)) - Z(g(X,Y)) \\ &- g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]) \end{aligned}$$

for all $X, Y, Z \in \mathfrak{X}(M)$. Furthermore, the Koszul formula **determines the connection uniquely**.

Proof: (see right-hand-side, alternatively exercise!)

Hence, we have shown that there exists precisely **one** torsion-free and metric connection in the tangent bundle of a given pseudo-Riemannian manifold. The Levi-Civita connection will be used in the development of the rest of the theory that we will study in this course.

In order to actually **calculate** with the Levi-Civita connection in local coordinates, we need to determine its **Christoffel symbols**:

Lemma

The **Christoffel symbols of the Levi-Civita connection** of a pseudo-Riemannian manifold (M, g) with respect to local coordinates $(x^1, \dots, x^n)$ are given by

$$\Gamma^k_{ij} = \frac{1}{2} \sum_{\ell=1}^{n} \left(\frac{\partial g_{j\ell}}{\partial x^i} + \frac{\partial g_{i\ell}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^\ell} \right) g^{\ell k}$$

for all $1 \leq i, j, k \leq n$.

Proof: Exercise!

Connections allow us to define a coordinate-free version of the **Hessian** and **Laplace operator**:

Definition

Let ∇ be a connection in $TM \to M$. The **covariant Hessian** of a smooth function $f \in C^\infty(M)$ is defined as the $(0,2)$-tensor field

$$\nabla^2 f := \nabla(\nabla f) = \nabla df \in \mathcal{T}^{0,2}(M).$$

If (M, g) is a pseudo-Riemannian manifold and ∇ is the Levi-Civita connection, we can take the trace of the covariant Hessian with respect to g and obtain the **Laplace-Beltrami operator** on smooth functions $f \in C^\infty(M)$ given by

$$\Delta f := \mathrm{tr}_g(\nabla^2 f).$$

Note: The covariant Hessian w.r.t. ∇ is **symmetric** if and only if ∇ is **torsion-free**. This in particular holds for the Levi-Civita connection.

END OF LECTURE 16

Next lecture:

- geodesics
- exponential map
- normal coordinates
- geodesic completeness
- Hopf-Rinow