Differential geometry
Lecture 15: Connections in vector bundles

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1 Motivation

2 Connections in vector bundles

Recap of lecture 14:

- introduces **(local) frames** of vector bundles
- described **subbundles** using local frames of ambient vector bundles, discussed **examples**
- defined **Killing vector fields**

Question 1: What is a good choice for a **derivative** of sections in vector bundles, in particular tensor powers of the tangent bundle?
Hint 1: Not the Lie derivative.

Question 2: What is a, in a sense preferred, way to **transport** vectors or, more generally, tensors in a given fibre to some other fibre in $TM \to M$, respectively $T^{r,s}M \to M$, along a piecewise smooth curve? What extra data do we need on M to make **our choice** the preferred choice?
Hint 2: Do not attempt to use a **coordinate-based** approach...

Question 3: Can we somehow **connect** fibres in a vector bundle in the sense that we can **identify** them via a preferred linear isomorphism for each pair of fibres?
Hint 3: If we can solve Q1 & Q2 and, frankly, work **a lot** more.

Definition

Let $E \to M$ be a vector bundle. A **connection in $E \to M$** is a bilinear map

$$\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E), \quad (X, s) \mapsto \nabla_X s,$$

that is **$C^\infty(M)$-linear in the first entry**, i.e.

$$\nabla_{fX} s = f \nabla_X s \quad \forall f \in C^\infty(M), X \in \mathfrak{X}(M), s \in \Gamma(E),$$

and fulfils the **Leibniz rule**

$$\nabla_X(fs) = X(f)s + f\nabla_X s \quad \forall f \in C^\infty(M), X \in \mathfrak{X}(M), s \in \Gamma(E).$$

The last condition can be written as $\nabla(fs) = s \otimes df + f\nabla s$.

Note: A connection can be canonically extended to be defined on **local sections**.

Remark

Recall that for $E = T^{r,s}M$, the Lie derivative is **not** $C^\infty(M)$-linear in the first entry of $(X, A) \mapsto \mathcal{L}_X A$, $X \in \mathfrak{X}(M)$, $A \in \mathcal{T}^{r,s}(M)$. Hence, $\mathcal{L}$ **is not a connection** in $T^{r,s}M \to M$ for any r, s.

Question: How do we actually calculate with a given connection in a vector bundle $E \to M$?
Answer: Use **local frames** of E and **local coordinates** on M.

Definition

Let ∇ be a connection in $E \to M$ of rank ℓ and $\{s_1, \ldots, s_\ell\}$ be a local frame over $U \subset M$ open, such that there exist local coordinates $(x^1, \ldots, x^n)$ on $U \subset M$. This can always be achieved after possibly shrinking U. Let further $\dim(M) = n$. Define

$$\nabla s_i := \omega_i, \quad \omega_i(X) = \nabla_X s_i \quad \forall X \in \mathfrak{X}(M),$$

for $1 \leq i \leq \ell$. Then each ω_i is an ***E*-valued 1-form**, that is $\omega_i \in \Gamma(E|_U \otimes T^*M|_U)$ for all $1 \leq i \leq \ell$. (continued on next page)

Definition (continuation)

Thus we have

$$\omega_i = \sum_{j=1}^{n} \omega_{ij} \otimes dx^j$$

for all $1 \leq i \leq \ell$, where $\omega_{ij} \in \Gamma(E|_U)$ for all $1 \leq i \leq \ell$, $1 \leq j \leq n$. We can further write

$$\omega_{ij} = \sum_{k=1}^{\ell} \omega_{ij}^k s_k,$$

with $\omega_{ij}^k \in C^\infty(U)$ for all $1 \leq i \leq \ell$, $1 \leq j \leq n$, $1 \leq k \leq \ell$. Recall that for any **local section** $s \in \Gamma(E|_U)$ we can write $s = \sum_{i=1}^{k} f^i s_i$ with f^i, $1 \leq i \leq k$, **uniquely determined** for s. We obtain the general formula

$$\nabla s = \sum_{i=1}^{\ell} s_i \otimes df^i + \sum_{j=1}^{n} \sum_{i,k=1}^{\ell} f^i \omega_{ij}^k s_k \otimes dx^j. \tag{1}$$

(continued on next page)

Definition (continuation)

On the other hand we might write

$$\nabla s_i = \omega_i = \sum_{k=1}^{\ell} s_k \otimes \omega_i^k$$

for all $1 \leq i \leq k$, where $\omega_i^k \in \Omega^1(U)$ for all $1 \leq i, k \leq \ell$. The ω_i^k are called **connection 1-forms** and determine the connection ∇ in $E|_U$ completely.We can view (ω_i^k) as an $(\ell \times \ell)$-**matrix valued map** where each entry is a **local 1-form** on M.

Warning: Even though connection 1-forms have "**1-form**" in their name, they **do not** transform like E-valued 1-forms when changing the **local frame** in E, e.g. via a change of coordinates on M when $E = T^{r,s}M$. (details on next page)
Question: Does every manifold **admit** a connection in its tangent bundle?
Answer: Yes! **[**Exercise! Alternatively, wait until we define the so-called Levi-Civita connection.**]**

Lemma

Let ∇ be a **connection** in a vector bundle $E \to M$ of rank ℓ. Let $\{s_1, \dots, s_\ell\}$ and $\{\widetilde{s}_1, \dots, \widetilde{s}_\ell\}$ be **local frames** of E **over a chart neighbourhood** $U \subset M$, equipped with local coordinates $(x^1, \dots, x^n)$, that are related by the $(\ell \times \ell)$**-matrix valued smooth map**

$$A : U \to \mathrm{GL}(\ell), \quad (s_1, \dots, s_\ell) \cdot A = (\widetilde{s}_1, \dots, \widetilde{s}_\ell).$$

Let (ω_i^k) denote the **matrix of connection 1-forms with respect to the local frame** $\{s_1, \dots, s_\ell\}$ and $(\widetilde{\omega}_i^k)$ the **matrix of connection 1-forms with respect to the local frame** $\{\widetilde{s}_1, \dots, \widetilde{s}_\ell\}$. Then

$$(\widetilde{\omega}_i^k) = A^{-1} dA + A^{-1} (\omega_i^k) A.$$

In the above equation, dA denotes the differential of the map $A : U \to \mathrm{GL}(\ell)$, where we identify $T\mathrm{GL}(\ell) \cong \mathrm{GL}(\ell) \times \mathrm{End}(\mathbb{R}^\ell)$.

Connections, like tangent vectors, are **local objects**:

Lemma A

Let ∇ be a connection in a vector bundle $E \to M$ of rank ℓ. Let $U \subset M$ be open and suppose that for two vector fields $X, Y \in \mathfrak{X}(M)$ and two sections in $E \to M$, $s, \widetilde{s}$, we have

$$X|_U = Y|_U, \quad s|_U = \widetilde{s}|_U.$$

Then $\nabla_X s$ and $\nabla_Y \widetilde{s}$ **coincide** on U.

Proof:

- $\nabla_X s|_U = \nabla_Y s|_U$ follows from the **tensoriality property** in the first argument of any connection
- $\rightsquigarrow$ suffices to show $\nabla_X s|_U = \nabla_X \widetilde{s}|_U$
- by the linearity in the second argument, $\nabla_X s$ and $\nabla_X \widetilde{s}$ coincide in U **if and only if** $\nabla_X (s - \widetilde{s})|_U \equiv 0$
- $\rightsquigarrow$ suffices to prove $\nabla_X s|_U = 0$ if $s|_U = 0$

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- **fix** $p \in U$, choose a **bump function** $b \in C^\infty(M)$ and an open nbh. of p, $V \subset U$, that is **precompact** in U, such that $b|_V \equiv 1$ and $\mathrm{supp}(b) \subset U$
- the **Leibniz rule** implies

$$0 = \nabla_X 0|_p = \nabla_X(bs)|_p = X(b)s|_p + b(p)\nabla_X s|_p = \nabla_X s|_p$$

- since $p \in U$ was **arbitrary**, above completes the proof □

Note: Lemma A means that $(\nabla_X s)(p)$ for any $p \in M$ depends only on $X_p \in T_pM$ and the restriction of s to an **arbitrary small** open neighbourhood of p in M

Example A

Consider $\mathbb{R}^n$ with **canonical coordinates** $(u^1, \ldots, u^n)$ and **induced global frame** $\left\{\frac{\partial}{\partial u^1}, \ldots, \frac{\partial}{\partial u^n}\right\}$ of $T\mathbb{R}^n \to \mathbb{R}^n$. Vector fields on $\mathbb{R}^n$ can be viewed as **smooth vector valued functions**. So a reasonable approach for a connection, **defined in our choice of coordinates**, is

$$\nabla_X Y := \sum_i X(Y^i) \frac{\partial}{\partial u^i} \in \mathfrak{X}(\mathbb{R}^n)$$

for all vector fields $X = \sum_i X^i \frac{\partial}{\partial u^i}$ and $Y = \sum_i Y^i \frac{\partial}{\partial u^i}$. This means that, in canonical coordinates, we **differentiate** Y **entrywise in** X**-direction**. One verifies that the so-defined ∇ **in fact is a connection** in $T\mathbb{R}^n \to \mathbb{R}^n$. This **construction** is, however, **not coordinate-independent**, meaning that in **different coordinates**, $\nabla_X Y$ will **not be the entrywise differentiation** of Y in X-direction. Note that all connection 1-forms of the above connection **identically vanish**.

Question: How does the connection in Example A look like in **different coordinates**, e.g. in polar coordinates?
Answer: To formalise this **type** of question, first define the following:

Definition

Let ∇ be a connection in $TM \to M$ and $(x^1, \dots, x^n)$ be local coordinates on $U \subset M$. Then in the induced local frame of TM,

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum_{k=1}^{n} \Gamma_{ij}^k \frac{\partial}{\partial x^k},$$

where $\Gamma_{ij}^k \in C^\infty(M)$, $1 \leq i, j, k \leq n$. The terms Γ_{ij}^k are called **Christoffel symbols** of the connection ∇ with respect to the chosen local coordinates $(x^1, \dots, x^n)$.

Note: The Christoffel symbols specify the connection ∇ in $TM|_U \to U$ completely, meaning in particular that two connections in $TM \to M$ coincide if they have the same Christoffel symbols for all local coordinates on M.

In comparison with the most general case, the Christoffel symbols are for the **special case** of the tangent bundle with induced local frame precisely the terms ω_{ij}^k on page 7.

Question: How do Christoffel symbols behave under a **change of coordinates**?
Answer:

Lemma

Let M be an n-dimensional smooth manifold, ∇ a connection in $TM \to M$. Let further $\varphi = (x^1, \dots, x^n)$ and $\psi = (y^n, \dots, y^n)$ local coordinate systems on an open set $U \subset M$. Let Γ_{ij}^k denote the **Christoffel symbols** of ∇ **with respect to** φ and $\widetilde{\Gamma}_{ij}^k$ denote the **Christoffel symbols** of ∇ **with respect to** ψ. Then the following **identity** holds:

$$\Gamma_{ij}^k = \sum_{\rho} \frac{\partial^2 y^\rho}{\partial x^i \partial x^j} \frac{\partial x^k}{\partial y^\rho} + \sum_{\mu,\nu,\rho} \frac{\partial y^\mu}{\partial x^i} \frac{\partial y^\nu}{\partial x^j} \frac{\partial x^k}{\partial y^\rho} \widetilde{\Gamma}_{\mu\nu}^\rho.$$

Proof: (on the right)

Example

In **polar coordinates** (r, φ), the connection ∇ in $T(\mathbb{R}^2 \setminus \{(x,0),\ x \leq 0\}) \to \mathbb{R}^2 \setminus \{(x,0),\ x \leq 0\}$ as in Example A has the following Christoffel symbols:

$$\Gamma^r_{\varphi\varphi} = -r, \quad \Gamma^\varphi_{r\varphi} = \Gamma^\varphi_{\varphi r} = \frac{1}{r}, \quad 0 \text{ else.}$$

Recall that knowing the Lie derivative on vector fields and functions allowed us to make sense of $\mathcal{L}_X A$ for $X \in \mathfrak{X}(M)$ and any type of tensor field $A \in \mathcal{T}^{r,s}(M)$, turning it into a **tensor derivative** for X fixed.

Question: Can we similarly define a **connection in** $T^{r,s}M \to M$ if we have a connection in the tangent bundle? Do we get a similar **Leibniz rule** for the tensor product?
Answer: Yes! (see next page)

Lemma

Let ∇ be a connection in $TM \to M$. Then ∇ induces a **connection ∇ in each tensor bundle** $T^{r,s}M \to M$, $r \geq 0$, $s \geq 0$, such that

- the induced connection in $T^{1,0}M \cong TM \to M$ **coincides with** ∇,
- $\nabla f = df$ for all $f \in \mathcal{T}^{0,0}(M) = C^\infty(M)$,
- the induced connection is a **tensor derivation** in the second argument, meaning that

$$\nabla(A \otimes B) = (\nabla A) \otimes B + A \otimes (\nabla B)$$

  whenever the tensor field $A \otimes B$ is defined,
- the induced connections **commute with all possible contraction**, meaning that for any contraction $C : \mathcal{T}^{r,s}(M) \to \mathcal{T}^{r-1,s-1}(M)$ we have

$$\nabla(C(A)) = C(\nabla(A))$$

  for all tensor fields $A \in \mathcal{T}^{r,s}(M)$.

Lemma (continuation)

The so-defined connections in each tensor bundle $T^{r,s}M \to M$ are **uniquely determined** by the above properties.

Proof:

- first define **candidates** for each connection, then we show that it **fulfils all requirements**, then prove **uniqueness**
- **note:** to define **any** connection in $T^{r,s}M \to M$ it suffices to specify what it does on sections that can be, locally, written as **pure tensor products** of r local vector fields and s local 1-forms
- for $T^{1,0}M \to M$, we simply take ∇ to be our **initial connection**, which thereby automatically fulfils the **first point**
- for $f \in \mathcal{T}^{0,0}(M) = C^\infty(M)$ we set $\nabla f = df$, thereby fulfilling the **second point**

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(continuation of proof)

- $\rightsquigarrow$ define ∇ in $T^{0,1}M \to M$ in such a way, that **the last two points** will be satisfied
- define for any local 1-form $\omega \in \Omega^1(U)$, $U \subset M$ open,

$$(\nabla_X \omega)(Y) := X(\omega(Y)) - \omega(\nabla_X Y) \quad \forall X, Y \in \mathfrak{X}(U)$$

- this in fact **defines a connection in** $T^{0,1}M \to M$
- $\rightsquigarrow$ obtain a connection in $T^{r,s}M \to M$ for all $r \geq 0$, $s \geq 0$ by requiring the thrid point, i.e. the tensor **Leibniz rule**, to hold on **pure** and, hence by **linear extension**, **on all tensor fields**
- again, this in fact defines a connection in each $T^{r,s}M \to M$
- $\rightsquigarrow$ remains to check that then the forth point, that is the **contraction property**, holds

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(continuation of proof)

- This can be done **inductively using** the third point, **the Leibniz rule**, after checking that it holds for the **only possible contraction in** $T^{1,1}M \to M$, which on pure tensor fields is of the form

$$C(X \otimes \omega) = \omega(X) \quad \forall X \in \mathfrak{X}(M), \omega \in \Omega^1(M)$$

- find for all $X, Y \in \mathfrak{X}(M)$ and all $\omega \in \Omega^1(M)$

$$\nabla_Y(C(X \otimes \omega)) = \nabla_Y(\omega(X)) = Y(\omega(X))$$

- **by definition** of ∇ in $T^{0,1}M \to M$ and the **imposed** third point (Leibniz), the above coincides with

$$Y(\omega(X)) = (\nabla_Y \omega)(X) + \omega(\nabla_Y X)$$
$$= C(X \otimes (\nabla_Y \omega) + (\nabla_Y X) \otimes \omega) = C(\nabla_Y(X \otimes \omega))$$

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(continuation of proof)

- remains to show **uniqueness** of so-defined connections
- suppose there is **an other connection** $\widetilde{\nabla}$ **fulfilling all requirements** of this lemma
- linearity in the second argument $\rightsquigarrow$ suffices to show that ∇ and $\widetilde{\nabla}$ coincide on **local pure tensor fields**
- **Leibniz rule** and $\nabla = \widetilde{\nabla}$ in $TM \to M$ $\rightsquigarrow$ suffices to show that ∇ and $\widetilde{\nabla}$ **coincide in** $T^{0,1}M = T^*M \to M$
- this follows from **first** ($\nabla = \widetilde{\nabla}$ in $TM \to M$), **second** ($\widetilde{\nabla} f = df$), and **forth** (contraction property) point by direct calculation of the left- and right-hand of $\widetilde{\nabla}(C(A)) = C(\widetilde{\nabla}(A))$ for $A = X \otimes \omega$ where X is **any local vector field** and ω is **any local 1-form** □

Remark

Differentiation of tensor fields with respect to a **connection** induced by a connection in the tangent bundle is sometimes called **covariant differentiation**. $\nabla_X A$ is then called **covariant derivative of** A **in direction** X.

END OF LECTURE 15

Next lecture:

- covariant differentiation along curves
- parallel transport
- torsion tensor of a connection
- metric connections
- Levi-Civita connection