Differential geometry
Lecture 15: Connections in vector bundles

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Motivation

Connections in vector bundles
Recap of lecture 14:

- introduces (local) frames of vector bundles
- described subbundles using local frames of ambient vector bundles, discussed examples
- defined Killing vector fields
**Question 1:** What is a good choice for a derivative of sections in vector bundles, in particular tensor powers of the tangent bundle?

**Hint 1:** Not the Lie derivative.

**Question 2:** What is a, in a sense preferred, way to transport vectors or, more generally, tensors in a given fibre to some other fibre in $TM \to M$, respectively $T^{r,s}M \to M$, along a piecewise smooth curve? What extra data do we need on $M$ to make our choice the preferred choice?

**Hint 2:** Do not attempt to use a coordinate-based approach...

**Question 3:** Can we somehow connect fibres in a vector bundle in the sense that we can identify them via a preferred linear isomorphism for each pair of fibres?

**Hint 3:** If we can solve Q1 & Q2 and, frankly, work a lot more.
Definition

Let $E \to M$ be a vector bundle. A **connection in** $E \to M$ is a bilinear map

$$\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E), \quad (X, s) \mapsto \nabla_X s,$$

that is $C^\infty(M)$-linear in the first entry, i.e.

$$\nabla_{fX} s = f \nabla_X s \quad \forall f \in C^\infty(M), X \in \mathfrak{X}(M), s \in \Gamma(E),$$

and fulfils the **Leibniz rule**

$$\nabla_X(fs) = X(f)s + f \nabla_X s \quad \forall f \in C^\infty(M), X \in \mathfrak{X}(M), s \in \Gamma(E).$$

The last condition can be written as $\nabla(fs) = s \otimes df + f \nabla s$.

**Note:** A connection can be canonically extended to be defined on **local sections**.
Remark

Recall that for $E = T^{r,s}M$, the Lie derivative is not $C^\infty(M)$-linear in the first entry of $(X, A) \mapsto \mathcal{L}_X A$, $X \in \mathfrak{X}(M)$, $A \in \mathcal{T}^{r,s}(M)$. Hence, $\mathcal{L}$ is not a connection in $T^{r,s}M \to M$ for any $r, s$.

Question: How do we actually calculate with a given connection in a vector bundle $E \to M$?

Answer: Use local frames of $E$ and local coordinates on $M$.

Definition

Let $\nabla$ be a connection in $E \to M$ of rank $\ell$ and $\{s_1, \ldots, s_\ell\}$ be a local frame over $U \subset M$ open, such that there exist local coordinates $(x^1, \ldots, x^n)$ on $U \subset M$. This can always be achieved after possibly shrinking $U$. Let further $\dim(M) = n$. Define

$$\nabla s_i := \omega_i, \quad \omega_i(X) = \nabla_X s_i \quad \forall X \in \mathfrak{X}(M),$$

for $1 \leq i \leq \ell$. Then each $\omega_i$ is an $E$-valued 1-form, that is $\omega_i \in \Gamma(E \big|_U \otimes T^*M \big|_U)$ for all $1 \leq i \leq \ell$. (continued on next page)
Thus we have
\[ \omega_i = \sum_{j=1}^{n} \omega_{ij} \otimes dx^j \]
for all \(1 \leq i \leq \ell\), where \(\omega_{ij} \in \Gamma(E|U)\) for all \(1 \leq i \leq \ell\), \(1 \leq j \leq n\). We can further write
\[ \omega_{ij} = \sum_{k=1}^{\ell} \omega^{k}_{ij} s_k, \]
with \(\omega^{k}_{ij} \in C^\infty(U)\) for all \(1 \leq i \leq \ell\), \(1 \leq j \leq n\), \(1 \leq k \leq \ell\).

Recall that for any local section \(s \in \Gamma(E|U)\) we can write
\[ s = \sum_{i=1}^{k} f^i s_i \]
with \(f^i\), \(1 \leq i \leq k\), uniquely determined for \(s\). We obtain the general formula
\[ \nabla s = \sum_{i=1}^{\ell} s_i \otimes df^i + \sum_{j=1}^{n} \sum_{i,k=1}^{\ell} f^i \omega^{k}_{ij} s_k \otimes dx^j. \] (1)

(continued on next page)
Definition (continuation)

On the other hand we might write

\[ \nabla s_i = \omega_i = \sum_{k=1}^{\ell} s_k \otimes \omega_i^k \]

for all \( 1 \leq i \leq k \), where \( \omega_i^k \in \Omega^1(U) \) for all \( 1 \leq i, k \leq \ell \). The \( \omega_i^k \) are called connection 1-forms and determine the connection \( \nabla \) in \( E|_U \) completely. We can view \( (\omega_i^k) \) as an \((\ell \times \ell)\)-matrix valued map where each entry is a local 1-form on \( M \).

**Warning:** Even though connection 1-forms have “1-form” in their name, they do not transform like \( E \)-valued 1-forms when changing the local frame in \( E \), e.g. via a change of coordinates on \( M \) when \( E = T^r|_M \). (details on next page)

**Question:** Does every manifold admit a connection in its tangent bundle?

**Answer:** Yes! [Exercise! Alternatively, wait until we define the so-called Levi-Civita connection.]
Lemma

Let $\nabla$ be a connection in a vector bundle $E \to M$ of rank $\ell$. Let $\{s_1, \ldots, s_\ell\}$ and $\{\tilde{s}_1, \ldots, \tilde{s}_\ell\}$ be local frames of $E$ over a chart neighbourhood $U \subset M$, equipped with local coordinates $(x^1, \ldots, x^n)$, that are related by the $(\ell \times \ell)$-matrix valued smooth map

$$A : U \to \text{GL}(\ell), \quad (s_1, \ldots, s_\ell) \cdot A = (\tilde{s}_1, \ldots, \tilde{s}_\ell).$$

Let $(\omega^k_i)$ denote the matrix of connection 1-forms with respect to the local frame $\{s_1, \ldots, s_\ell\}$ and $(\tilde{\omega}^k_i)$ the matrix of connection 1-forms with respect to the local frame $\{\tilde{s}_1, \ldots, \tilde{s}_\ell\}$. Then

$$(\tilde{\omega}^k_i) = A^{-1} dA + A^{-1} (\omega^k_i) A.$$

In the above equation, $dA$ denotes the differential of the map $A : U \to \text{GL}(\ell)$, where we identify $T\text{GL}(\ell) \cong \text{GL}(\ell) \times \text{End}(\mathbb{R}^\ell)$. 

Connections, like tangent vectors, are **local objects**: 

**Lemma A**

Let $\nabla$ be a connection in a vector bundle $E \to M$ of rank $\ell$. Let $U \subset M$ be open and suppose that for two vector fields $X, Y \in \mathfrak{X}(M)$ and two sections in $E \to M$, $s, \tilde{s}$, we have

$$X|_U = Y|_U, \quad s|_U = \tilde{s}|_U.$$  

Then $\nabla_X s$ and $\nabla_Y \tilde{s}$ **coincide** on $U$.

**Proof:**

- $\nabla_X s|_U = \nabla_Y s|_U$ follows from the **tensoriality property** in the first argument of any connection.
- $\leftrightarrow$ suffices to show $\nabla_X s|_U = \nabla_X \tilde{s}|_U$
- by the linearity in the second argument, $\nabla_X s$ and $\nabla_X \tilde{s}$ coincide in $U$ if and only if $\nabla_X (s - \tilde{s})|_U \equiv 0$
- $\leftrightarrow$ suffices to prove $\nabla_X s|_U = 0$ if $s|_U = 0$

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**Connections in vector bundles**

- fix $p \in U$, choose a **bump function** $b \in C^\infty(M)$ and an open nbh. of $p$, $V \subset U$, that is **precompact** in $U$, such that $b|_V \equiv 1$ and $\text{supp}(b) \subset U$

- the **Leibniz rule** implies

  $$0 = \nabla_X 0|_p = \nabla_X (bs)|_p = X(b)s|_p + b(p) \nabla_X s|_p = \nabla_X s|_p$$

- since $p \in U$ was **arbitrary**, above completes the proof

**Note:** Lemma A means that $(\nabla_X s)(p)$ for any $p \in M$ depends only on $X_p \in T_p M$ and the restriction of $s$ to an **arbitrary small** open neighbourhood of $p$ in $M$
Consider $\mathbb{R}^n$ with canonical coordinates $(u^1, \ldots, u^n)$ and induced global frame $\{\frac{\partial}{\partial u^1}, \ldots, \frac{\partial}{\partial u^n}\}$ of $T\mathbb{R}^n \to \mathbb{R}^n$. Vector fields on $\mathbb{R}^n$ can be viewed as smooth vector valued functions. So a reasonable approach for a connection, defined in our choice of coordinates, is

$$\nabla_X Y := \sum_i X(Y^i) \frac{\partial}{\partial u^i} \in \mathfrak{X}(\mathbb{R}^n)$$

for all vector fields $X = \sum_i X^i \frac{\partial}{\partial u^i}$ and $Y = \sum_i Y^i \frac{\partial}{\partial u^i}$. This means that, in canonical coordinates, we differentiate $Y$ entrywise in $X$-direction. One verifies that the so-defined $\nabla$ in fact is a connection in $T\mathbb{R}^n \to \mathbb{R}^n$. This construction is, however, not coordinate-independent, meaning that in different coordinates, $\nabla_X Y$ will not be the entrywise differentiation of $Y$ in $X$-direction. Note that all connection 1-forms of the above connection identically vanish.
**Question:** How does the connection in Example A look like in different coordinates, e.g. in polar coordinates?

**Answer:** To formalise this type of question, first define the following:

**Definition**

Let $\nabla$ be a connection in $TM \to M$ and $(x^1, \ldots, x^n)$ be local coordinates on $U \subset M$. Then in the induced local frame of $TM$,

$$\nabla \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} = \sum_{k=1}^{n} \Gamma_{ij}^k \frac{\partial}{\partial x^k},$$

where $\Gamma_{ij}^k \in C^\infty(M)$, $1 \leq i, j, k \leq n$. The terms $\Gamma_{ij}^k$ are called Christoffel symbols of the connection $\nabla$ with respect to the chosen local coordinates $(x^1, \ldots, x^n)$.

**Note:** The Christoffel symbols specify the connection $\nabla$ in $TM|_U \to U$ completely, meaning in particular that two connections in $TM \to M$ coincide if they have the same Christoffel symbols for all local coordinates on $M$. 
In comparison with the most general case, the Christoffel symbols are for the special case of the tangent bundle with induced local frame precisely the terms $\omega^k_{ij}$ on page 7.

**Question:** How do Christoffel symbols behave under a change of coordinates?

**Answer:**

**Lemma**

Let $M$ be an $n$-dimensional smooth manifold, $\nabla$ a connection in $TM \to M$. Let further $\varphi = (x^1, \ldots, x^n)$ and $\psi = (y^n, \ldots, y^n)$ local coordinate systems on an open set $U \subset M$. Let $\Gamma^k_{ij}$ denote the Christoffel symbols of $\nabla$ with respect to $\varphi$ and $\tilde{\Gamma}^k_{ij}$ denote the Christoffel symbols of $\nabla$ with respect to $\psi$. Then the following identity holds:

$$\Gamma^k_{ij} = \sum_{\rho} \frac{\partial^2 y^\rho}{\partial x^i \partial x^j} \frac{\partial x^k}{\partial y^\rho} + \sum_{\mu, \nu, \rho} \frac{\partial y^\mu}{\partial x^i} \frac{\partial y^\nu}{\partial x^j} \frac{\partial x^k}{\partial y^\rho} \tilde{\Gamma}^\rho_{\mu \nu}. $$

**Proof:** (on the right)
Example

In polar coordinates \((r, \varphi)\), the connection \(\nabla\) in \(T(\mathbb{R}^2 \setminus \{(x, 0), \ x \leq 0\}) \rightarrow \mathbb{R}^2 \setminus \{(x, 0), \ x \leq 0\}\) as in Example A has the following Christoffel symbols:

\[
\Gamma^r_{\varphi\varphi} = -r, \quad \Gamma^\varphi_{r\varphi} = \Gamma^\varphi_{\varphi r} = \frac{1}{r}, \quad 0 \text{ else.}
\]

Recall that knowing the Lie derivative on vector fields and functions allowed us to make sense of \(\mathcal{L}_X A\) for \(X \in \mathfrak{X}(M)\) and any type of tensor field \(A \in \mathcal{T}^{r,s}(M)\), turning it into a tensor derivative for \(X\) fixed.

**Question:** Can we similarly define a **connection in** \(\mathcal{T}^{r,s} M \rightarrow M\) if we have a connection in the tangent bundle? Do we get a similar **Leibniz rule** for the tensor product?  
**Answer:** Yes! (see next page)
Lemma

Let $\nabla$ be a connection in $TM \to M$. Then $\nabla$ induces a connection $\nabla$ in each tensor bundle $T^{r,s}M \to M$, $r \geq 0$, $s \geq 0$, such that

- the induced connection in $T^{1,0}M \cong TM \to M$ coincides with $\nabla$,
- $\nabla f = df$ for all $f \in \mathcal{C}^{0,0}(M) = \mathcal{C}^{\infty}(M)$,
- the induced connection is a tensor derivation in the second argument, meaning that
  \[
  \nabla(A \otimes B) = (\nabla A) \otimes B + A \otimes (\nabla B)
  \]
  whenever the tensor field $A \otimes B$ is defined,
- the induced connections commute with all possible contraction, meaning that for any contraction $C : \mathcal{T}^{r,s}(M) \to \mathcal{T}^{r-1,s-1}(M)$ we have
  \[
  \nabla(C(A)) = C(\nabla(A))
  \]
  for all tensor fields $A \in \mathcal{T}^{r,s}(M)$. 
Lemma (continuation)

The so-defined connections in each tensor bundle $T^{r,s}M \to M$ are **uniquely determined** by the above properties.

Proof:

- first define **candidates** for each connection, then we show that it **fulfils all requirements**, then prove **uniqueness**
- **note:** to define any connection in $T^{r,s}M \to M$ it suffices to specify what it does on sections that can be, locally, written as **pure tensor products** of $r$ local vector fields and $s$ local 1-forms
- for $T^{1,0}M \to M$, we simply take $\nabla$ to be our **initial connection**, which thereby automatically fulfils the **first point**
- for $f \in \mathcal{C}^{0,0}(M) = C^{\infty}(M)$ we set $\nabla f = df$, thereby fulfilling the **second point**

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(continuation of proof)

- Define $\nabla$ in $T^{0,1}M \to M$ in such a way, that the last two points will be satisfied.
- Define for any local 1-form $\omega \in \Omega^1(U), U \subset M$ open,
  \[
  (\nabla_X \omega)(Y) := X(\omega(Y)) - \omega(\nabla_X Y) \quad \forall X, Y \in \mathfrak{X}(U)
  \]

- This in fact defines a connection in $T^{0,1}M \to M$.
- Obtain a connection in $T^{r,s}M \to M$ for all $r \geq 0, s \geq 0$ by requiring the third point, i.e. the tensor Leibniz rule, to hold on pure and, hence by linear extension, on all tensor fields.
- Again, this in fact defines a connection in each $T^{r,s}M \to M$.
- Remains to check that then the forth point, that is the contraction property, holds.

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(continuation of proof)

- This can be done **inductively using** the third point, the **Leibniz rule**, after checking that it holds for the **only possible contraction in** $T^{1,1}M \to M$, which on pure tensor fields is of the form

$$C(X \otimes \omega) = \omega(X) \quad \forall X \in \mathfrak{X}(M), \omega \in \Omega^1(M)$$

- find for all $X, Y \in \mathfrak{X}(M)$ and all $\omega \in \Omega^1(M)$

$$\nabla_Y(C(X \otimes \omega)) = \nabla_Y(\omega(X)) = Y(\omega(X))$$

- **by definition** of $\nabla$ in $T^{0,1}M \to M$ and the **imposed** third point (Leibniz), the above coincides with

$$Y(\omega(X)) = (\nabla_Y \omega)(X) + \omega(\nabla_Y X)$$
$$= C(X \otimes (\nabla_Y \omega) + (\nabla_Y X) \otimes \omega) = C(\nabla_Y (X \otimes \omega))$$

(continued on next page)
remains to show \textbf{uniqueness} of so-defined connections

suppose there is \textbf{an other connection} $\tilde{\nabla}$ fulfilling all \textbf{requirements} of this lemma

linearity in the second argument $\rightsquigarrow$ suffices to show that $\nabla$ and $\tilde{\nabla}$ coincide on \textbf{local pure tensor fields}

\textbf{Leibniz rule} and $\nabla = \tilde{\nabla}$ in $TM \to M$ $\rightsquigarrow$ suffices to show that $\nabla$ and $\tilde{\nabla}$ coincide in $T^{0,1}M = T^*M \to M$

this follows from \textbf{first} ($\nabla = \tilde{\nabla}$ in $TM \to M$), \textbf{second} ($\tilde{\nabla}f = df$), and \textbf{forth} (contraction property) point by direct calculation of the left- and right-hand of $\tilde{\nabla}(C(A)) = C(\tilde{\nabla}(A))$ for $A = X \otimes \omega$ where $X$ is \textbf{any local vector field} and $\omega$ is \textbf{any local 1-form}
Remark

Differentiation of tensor fields with respect to a connection induced by a connection in the tangent bundle is sometimes called covariant differentiation. $\nabla_X A$ is then called covariant derivative of $A$ in direction $X$. 
END OF LECTURE 15

Next lecture:

- covariant differentiation along curves
- parallel transport
- torsion tensor of a connection
- metric connections
- Levi-Civita connection