Differential geometry
Lecture 14: Local frames and Killing vector fields

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Recap of lecture 13:

- defined **trace** and **induced scalar product** in tensor bundles
- explained how to **raise** and **lower** indices of tensor fields
- defined vector bundles **along** submanifolds, in particular **tangent** and **orthogonal bundle** for pseudo-Riemannian submanifolds
- **erratum:** called the position vector field "tangent" to $H^n_\nu \subset \mathbb{R}^{n+1}$, correct would have been "**orthogonal**"

We know what a **basis of a vector space** is, and in the example of the tensor bundles of a smooth manifold $T^{r,s}M \to M$ how to **fibrewise** obtain a basis of $T_p^{r,s}M$ via a choice of local coordinates near $p \in M$.

Question: What is the correct local (**not** just fibrewise) setting for choosing bases of fibres over an open subset of the base space of vector bundles, and how does it fit in with what we already learned?
Answer: Define, locally, for each fibre a basis that **varies smoothly**.

Definition

Let $E \to M$ be a vector bundle of rank k. A **(local) frame of** E **over** $U \subset M$, U open, is a set of k **(local) sections**

$$\{s_i \in \Gamma(E|_U),\ 1 \leq i \leq k\},$$

such that for all $p \in U$ fixed, the vectors $s_i(p) \in E_p$, $1 \leq i \leq k$, are **linearly independent**. Equivalently,

$$\mathrm{span}_{\mathbb{R}}\{s_i(p) \in E_p \mid 1 \leq i \leq k\} = E_p \quad \forall p \in U.$$

Examples

- A local frame in $TM \to M$ over $U \subset M$ is a set of $n = \dim(M)$ **local vector fields** $\{X_1, \dots, X_n \in \mathfrak{X}(U)\}$ such that, pointwise, $X_{1p}, \dots, X_{np} \in T_pM$ are linearly independent.
- Local coordinates $(x^1, \dots, x^n)$ on $U \subset M$ induce the local frame $\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\}$ of **local coordinate vector fields** in $TM \to M$ over U.
- Similarly, the **local coordinate 1-forms** $\{dx^1, \dots, dx^n\}$ are a frame of $T^*M \to M$ over U.

Local trivializations can be constructed from local frames:

Exercise

Suppose that you are given a **local frame** of $E \to M$ over $U \subset M$. Construct a **local trivialization** of $E \to M$ using this data.

Note: Every local section $s \in \Gamma(E|_U)$ can be written as a $C^\infty(U)$-**linear combination** of the elements of a local frame of $E \to M$ over U. The prefactors are **uniquely determined** for a given local section.
Local frames are a nice tool in order to check if a **fibrewise linear subspace** $F \subset E$ of a vector bundle $E \to M$ is a subbundle:

Lemma A

Let $E \to M$ be a vector bundle of rank k and suppose that for $\ell \leq k$ we are given a **linear subspace** $F_p \subset E_p$ of constant dimension ℓ for all $p \in M$. Then $\bigsqcup_{p \in M} F_p \to M$ is, with all data necessary induced by $E \to M$, a **subbundle of** $E \to M$ if and only if for every $p \in M$ we can **find a local frame** $\{s_1, \ldots, s_k\}$ of $E|_U \to U$, $U \subset M$ an open neighbourhood of p, **such that for all** $q \in U$**,** $\{s_1(q), \ldots, s_\ell(q)\}$ **is a basis of** F_q.

Proof: See Lem. 10.32 *Riemannian Manifolds – An Introduction to Curvature*, Springer GTM 176, by John M. Lee. □

Lemma A implies the following for the **local form** of subbundles:

Corollary

Let $F \to M$ be a subbundle of rank ℓ of a vector bundle $E \to M$ of rank $k > \ell$. For any $p \in M$ we can find an **open neighbourhood** $U \subset M$ **of** p and a **local trivialization of** $E \to M$ **over** U, $\phi : E|_U \to U \times \mathbb{R}^k$, such that

$$\phi(\iota(F|_U)) =$$
$$U \times \{(v^1, \ldots, v^\ell, 0, \ldots, 0) \mid (v^1, \ldots, v^\ell) \in \mathbb{R}^\ell\} \subset U \times \mathbb{R}^k,$$

where ι denotes the inclusion.

Using the local frames, we will next describe two prominent **subbundles** of the $(0,2)$-**tensor bundle** of a smooth manifold. First, recall the following construction from linear algebra:

Lemma

Let V be a **finite-dimensional real vector space** with basis $\{v_1, \dots, v_n\}$. Then

$$V \otimes V \cong \mathrm{Sym}^2(V) \oplus \Lambda^2 V,$$

where $\mathrm{Sym}^2(V) := \mathrm{span}_{\mathbb{R}}\{v_i \otimes v_j + v_j \otimes v_i,\ 1 \leq i,j \leq n\}$ and $\Lambda^2 V := \mathrm{span}_{\mathbb{R}}\{v_i \otimes v_j - v_j \otimes v_i,\ 1 \leq i,j \leq n\}$.

Notation: $v_i v_j := \frac{1}{2}(v_i \otimes v_j + v_j \otimes v_i)$, $\quad v_i \wedge v_j := v_i \otimes v_j - v_j \otimes v_i$, so that $v_i \otimes v_j = v_i v_j + \frac{1}{2} v_i \wedge v_j$ and $v_i \otimes v_i = v_i v_i$ for all i,j

Question: How do we formulate a similar statement for $T^{0,2}M \to M$?

Answer: Fibrewise using local frames! (next page)

Definition

Let M be a smooth manifold and $(x^1, \dots, x^n)$ be local coordinates on $U \subset M$. The **bundle of symmetric** $(0,2)$**-tensors** on M is the subbundle

$$\mathrm{Sym}^2(T^*M) \subset T^{0,2}M$$

with local frame over U given by $\left\{dx^i dx^j = \frac{1}{2}(dx^i \otimes dx^j + dx^j \otimes dx^i),\ 1 \leq i,j \leq n\right\}$. Sections of $\mathrm{Sym}^2(T^*M)$ are precisely **symmetric** $(0,2)$**-tensor fields**, which in particular include all possible pseudo-Riemannian metrics on M. On the other hand we have the **anti-symmetric** $(0,2)$**-tensors** on M,

$$\Lambda^2 T^*M \subset T^{0,2}M,$$

with local frame over U given by $\left\{dx^i \wedge dx^j = dx^i \otimes dx^j - dx^j \otimes dx^i,\ 1 \leq i,j \leq n\right\}$. Sections in $\Lambda^2 T^*M \to M$ are called **2-forms** and are denoted by $\Omega^2(M)$. Local sections in $\Lambda^2 T^*M \to M$ over $U \subset M$, U open, are denoted by $\Omega^2(U)$.

Remark

Any pseudo-Riemannian metric g on M can be written **locally** as

$$g = \sum_{i,j} g_{ij} dx^i \otimes dx^j = \sum_{i,j} g_{ij} dx^i dx^j.$$

Warning: The above notation is standard, but has a certain error potential. Make **absolutely** sure to e.g. understand the equality

$$dx\, dy \text{ “}=\text{” } \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix},$$

where in the above equation $dx\, dy$ is written on the right hand side in (**in actual calculations commonly used**) matrix notation.

Suppose you are given a symmetric, fibrewise nondegenerate, $(0,2)$-tensor field on a connected manifold.
Question: How do you, realistically, **check** whether it is a pseudo-Riemannian metric or not?
Answer: Check that its **index** is constant!

Definition

The **index** of a symmetric $(0,2)$-tensor field $g \in \mathcal{T}^{0,2}(M)$ at $p \in M$ is defined as

$$\nu(p) := \text{ number of negative eigenvalues of } g_p,$$

where g_p is viewed as symmetric matrix in local coordinates, i.e.

$$g_p = \sum_{ij} g_{ij}(p) dx^i \otimes dx^j.$$

Proposition

Let M be a connected smooth manifold and $g \in \mathcal{T}^{0,2}(M)$ a **symmetric** $(0,2)$-**tensor field** that is **nondegenerate in all fibres** T_pM**,** $p \in M$. Then g is a **pseudo-Riemannian metric**.

Proof:

- suffices to show that the index of g, $\nu : M \to \mathbb{N}_0$, is **continuous**
- for this it suffices to prove that the number of negative eigenvalues of any smooth function with values in the symmetric $n \times n$-matrices,

$$A : I \to \mathrm{Sym}^2((\mathbb{R}^*)^n), \quad t \mapsto A(t) \in \mathrm{Sym}^2((\mathbb{R}^*)^n),$$

  such that $A(t)$ is nondegenerate for all $t \in I$, is **locally constant**
- this follows from the **continuity of the eigenvalues** of $A(t)$ viewed each as functions of t

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(continuation of proof)

- **[**more precisely: There exists a choice of n nowhere vanishing continuous functions $\lambda_n : I \to \mathbb{R} \setminus \{0\}$, such that for all $t \in I$, the set $\{(1, \lambda_1(t)), \dots, (n, \lambda_n(t))\}$ is precisely the set of (indexed) eigenvalues of $A(t)$**]**
- consider the **characteristic polynomial** of $A(t)$ **in dependence of** $t \in I$,

$$P_t(\lambda) := \det(A(t) - \lambda \mathbb{1})$$

- $P_t(\lambda)$ is of the **form**

$$P_t(\lambda) = \sum_{i=0}^{n} a_i(t) \lambda^i,$$

where $a_i : I \to \mathbb{R}$ is smooth for all $0 \leq i \leq n$ and $a_n(t) \equiv (-1)^n$

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(continuation of proof)

- hence: suffices to prove **continuous dependence of roots of a polynomial** of fixed degree with smoothly varying prefactors and fixed highest order monomial
- we know that the eigenvalues must be **real** by the **symmetry condition** of $A(t)$ and can use the main result in *Continuity and Location of Zeroes of Linear Combinations of Polynomials* (M. Zedek), Proc. Amer. Math. Soc. **16** (1965) $\square$

Recall the definition of **isometries** of pseudo-Riemannian manifolds and note that the **identity map** is an isometry in any case.
Question: How can we describe isometries **infinitesimally**, as in **infinitesimal perturbations of the identity**?
Answer: Use **local flows** and the **Lie derivative of tensor fields**!

Proposition

Let (M, g) be a pseudo-Riemannian manifold and let $X \in \mathfrak{X}(M)$. Suppose that for **every local flow** $\varphi : I \times U \to M$ of X, $\varphi_t : U \to M$ is an **isometry** for all $t \in I$. Then $\mathcal{L}_X g = 0$. The **converse statement** also holds true.

Proof:

- "$\Rightarrow$": a local flow $\varphi : I \times U \to M$ of X is an **isometry** of (M, g) for all $t \in I$ if and only if

$$g_p(v, w) = g_{\varphi_t(p)}(d\varphi_t(v), d\varphi_t(w))$$

  for all $t \in I$, $p \in M$, $v, w \in T_pM$

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(continuation of proof)

- hence:

$$
\begin{aligned}
(\mathcal{L}_X g)(v,w) &= \left(\frac{\partial}{\partial t}\bigg|_{t=0} (\varphi_t^* g)_p \right)(v,w) \\
&= \frac{\partial}{\partial t}\bigg|_{t=0} \left(g_{\varphi_t(p)}(d\varphi_t(v), d\varphi_t(w)) \right) \\
&= \frac{\partial}{\partial t}\bigg|_{t=0} g_p(v,w) = 0
\end{aligned}
$$

- since $p \in M$, $v, w \in T_pM$ were **arbitrary**, this shows that $\mathcal{L}_X g = 0$
- "$\Leftarrow$": **note:** $d\varphi_{t_0} : T_pM \to T_{\varphi_{t_0}(p)}M$ is a **linear isomorphism** for all $t_0 \in I$ and, by the **group property** of local flows, that $d\varphi_{t+t_0} = d\varphi_t d\varphi_{t_0}$ for t small enough

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(continuation of proof)

- we obtain $\forall t_0 \in I, v, w \in T_pM$

$$
\begin{aligned}
0 &= (\mathcal{L}_X g)(d\varphi_{t_0}(v), d\varphi_{t_0}(w)) \\
&= \frac{\partial}{\partial t}\bigg|_{t=0} \Big(g_{\varphi_t(\varphi_{t_0}(p))}(d\varphi_t d\varphi_{t_0}(v), d\varphi_t d\varphi_{t_0}(w)) \Big) \\
&= \frac{\partial}{\partial t}\bigg|_{t=0} \Big(g_{\varphi_{t+t_0}(p)}(d\varphi_{t+t_0}(v), d\varphi_{t+t_0}(w)) \Big) \\
&= \frac{\partial}{\partial s}\bigg|_{s=t_0} \big(g_{\varphi_s(p)}(d\varphi_s(v), d\varphi_s(w)) \big)
\end{aligned}
$$

- this shows that the smooth function

$$
I \ni s \mapsto g_{\varphi_s(p)}(d\varphi_s(v), d\varphi_s(w)) \in \mathbb{R}
$$

is **constant** for all $v, w \in T_pM$ and, hence, that the local flow of X **consists of isometries for any fixed time parameter** $\square$

Definition

Vector fields $X \in \mathfrak{X}(M)$ on a pseudo-Riemannian manifold (M, g) with $\mathcal{L}_X g = 0$ are called **Killing vector fields**.

The set of Killing vector fields on a pseudo-Riemannian manifold has the following algebraic structure:

Lemma

Let (M, g) be a pseudo-Riemannian manifold. Killing vector fields form a **Lie subalgebra** of $(\mathfrak{X}(M), [\cdot, \cdot])$, meaning that for any Killing vector fields $X, Y \in \mathfrak{X}(M)$, $[X, Y]$ is also a Killing vector field.

Proof: Exercise! **[**Hint: Use the Jacobi identity $\mathcal{L}_{[X,Y]} Z = \mathcal{L}_X(\mathcal{L}_Y Z) - \mathcal{L}_Y(\mathcal{L}_X Z)$**]**

One can prove the following theorem about a dimensional bound of the Lie algebra of Killing vector fields, but the proof goes far beyond the scope of this course, cf. Thm 3.3 in *Foundations of Differential Geometry* **Vol. I** (S. Kobayashi, K. Nomizu), Wiley Classics Library (1996)

Theorem

Let (M, g) be a **connected Riemannian manifold** of dimension n. The the Lie algebra of Killing vector fields is **finite dimensional of dimension at most** $\frac{1}{2}n(n+1)$.

Next, let us look at some **examples** of Killing vector fields. (next page)

Examples

- Consider $(M,g) = (\mathbb{R}^n, \langle\cdot,\cdot\rangle_\nu)$ for any $0 \leq \nu \leq n$. Then $X \in \mathfrak{X}(\mathbb{R}^n)$, $X = \sum_i c^i \frac{\partial}{\partial u^i}$, is a Killing vector field.
- Let (M,g) and (N,h) be pseudo-Riemannian manifolds, X a Killing vector field on (M,g), and Y a Killing vector field on (N,h). Then $X+Y$ is a Killing vector field on $(M \times N, g \oplus h)$.

Question: How can we **determine** Killing vector fields if we are not magically presented with them?
Answer: In **local coordinates**, have the following result:

Lemma

$X \in \mathfrak{X}(M)$ on a pseudo-Riemannian manifold (M,g) is a Killing vector field **if and and only if** it fulfils

$$\sum_{k=1}^{n} \left(X^k \frac{\partial g_{ij}}{\partial x^k} + \frac{\partial X^k}{\partial x^i} g_{jk} + \frac{\partial X^k}{\partial x^j} g_{ik} \right) = 0 \quad \forall 1 \leq i,j \leq n$$

for **all local coordinates** $(x^1, \ldots, x^n)$ on M.

Proof: Exercise!

END OF LECTURE 14

Next lecture:

- connections in vector bundles