Differential geometry
Lecture 13: Traces, raising and lowering indices, and vector bundles along pseudo-Riemannian submanifolds

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Recap of lecture 12:

- recalled definition of **pseudo-Euclidean vector spaces**
- defined **pseudo-Riemannian metrics** & **pseudo-Riemannian manifolds**
- defined **arc-length** of curves
- described pseudo-Riemannian metrics **in local coordinates**
- studied **examples** of pseudo-Riemannian manifolds, in particular **Riemannian submanifolds of Riemannian manifolds**

Recall the definition of the trace from linear algebra:

Definition

Let V be a real finite-dimensional vector space and $A \in \mathrm{End}(V) \cong V \otimes V^*$, so that for a basis $\{v_1, \ldots, v_n\}$ of V

$$A = \sum_{i,j=1}^{n} a^i{}_j v_i \otimes v_j^*.$$

The **trace** of A is defined as

$$\mathrm{tr}(A) := \sum_{i=1}^{n} a^i{}_i.$$

Note: The trace of an endomorphism is **independent** of the chosen basis.

Examples

- $\mathrm{tr}(\mathrm{id}_V) = \dim(V)$,
- $\mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}(B)$ for all $A, B \in \mathrm{End}(V)$,
- $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for all $A, B \in \mathrm{End}(V)$,
- $\mathrm{tr}(v \otimes \omega) = \omega(v)$ for all $v \in V$, $\omega \in V^*$.
- if $A : I \to \mathrm{GL}(n)$ is a **smooth curve**, we have $\frac{\partial}{\partial t} \det(A) = \mathrm{tr}\left(A^{-1} \frac{\partial A}{\partial t}\right) \det(A)$

If $(V, \langle\cdot,\cdot\rangle)$ is a pseudo-Euclidean vector space, we can calculate the trace as follows:

Lemma

Let $A \in \mathrm{End}(V)$ and $\{e_1, \ldots, e_n\}$ be an **orthonormal basis** of V w.r.t. $\langle\cdot,\cdot\rangle$. Then

$$\mathrm{tr}(A) = \sum_{i=1}^{n} \varepsilon_i \langle e_i, Ae_i \rangle,$$

where $\varepsilon_i := \langle e_i, e_i \rangle \in \{-1, 1\}$ for all $1 \leq i \leq n$.

If we have a pseudo-Euclidean vector space $(V, \langle\cdot,\cdot\rangle)$, we can define a scalar product on tensor powers of V and V^*.

Definition

Let $\{e_1, \dots, e_n\}$ be a basis of $(V, \langle\cdot,\cdot\rangle)$, $A \in V^{\otimes r} \otimes (V^*)^{\otimes s}$, and write $\langle\cdot,\cdot\rangle = \sum_{i,j=1}^{n} g_{ij} e_i^* \otimes e_j^*$,

$$A = \sum_{\substack{1 \le i_1, \dots, i_r \le n \\ 1 \le j_1, \dots, j_r \le n}} A^{i_1 \dots i_r}{}_{j_1 \dots j_s} e_{i_1} \otimes \dots \otimes e_{i_r} \otimes e_{j_1}^* \otimes \dots \otimes e_{j_s}^*.$$

Then

$\langle A, A \rangle :=$

$$\sum_{\substack{1 \le i_1, \dots, i_r \le n \\ 1 \le j_1, \dots, j_r \le n \\ 1 \le I_1, \dots, I_r \le n \\ 1 \le J_1, \dots, J_r \le n}} A^{i_1 \dots i_r}{}_{j_1 \dots j_s} A^{I_1 \dots I_r}{}_{J_1 \dots J_s} g_{i_1 I_1} \cdot \dots \cdot g_{i_r I_r} \cdot g^{j_1 J_1} \cdot \dots \cdot g^{j_s J_s}$$

defines a **possibly indefinite symmetric bilinear form** on $V^{\otimes r} \otimes (V^*)^{\otimes s}$. In the above formula the g-terms fulfil, when viewed as a symmetric matrix, $(g^{ij}) := (g_{ij})^{-1}$.

By switching from **pseudo-Euclidean vector spaces** to **pseudo-Riemannian manifolds**, we get the following fibrewise analogous definition:

Definition

Let (M, g) be a pseudo-Riemannian manifold, $A \in \mathcal{T}^{1,1}(M)$ an endomorphism field, $h \in \mathcal{T}^{0,2}(M)$ a symmetric $(0,2)$-tensor field, and $B \in \mathcal{T}^{r,s}(M)$ for $r + s > 0$ an arbitrary tensor field. Then the **trace of** A is in local coordinates $(x^1, \dots, x^n)$, so that $A = \sum A^i{}_j \frac{\partial}{\partial x^i} \otimes dx^j$, given by

$$\mathrm{tr}(A) := \sum_i A^i{}_i.$$

The above term is **invariant under coordinate change**, which follows from fibrewise invariance of the choice of basis in T_pM and the fact that the coordinate cotangent vector at each point are precisely the **dual** to the coordinate tangent vectors at that point. This means that $\mathrm{tr}_g(A) \in C^\infty(M)$.
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Definition (continuation)

The trace of h with respect to g is defined in local coordinates as

$$\mathrm{tr}_g(h) := \sum_{i,j} h_{ij} g^{ij}.$$

As for the endomorphism field, $\mathrm{tr}_g(h)$ is **invariant under coordinate change**, but **not** invariant of the pseudo-Riemannian metric g. Furthermore, we define the induced pairing of B with itself with respect to g in the given local coordinates as

$$g(B,B) := \sum B^{i_1 \dots i_r}{}_{j_1 \dots j_s} \cdot B^{I_1 \dots I_r}{}_{J_1 \dots J_s} \cdot g_{i_1 I_1} \cdot \dots \cdot g_{i_r I_r} \cdot g^{j_1 J_1} \cdot \dots \cdot g^{j_s J_s},$$

where

$$B = \sum B^{i_1 \dots i_r}{}_{j_1 \dots j_s} \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_s}$$

and $(g^{ij}) = (g_{ij})^{-1}$ at each point when viewed as a symmetric matrix valued map. As for the trace, value of $g(B,B)$ **does not depend on the choice of local coordinates** which implies $g(B,B) \in C^\infty(M)$.

Note: One can similarly define a symmetric pairing g in the bundle $T^{r,s}M \to M$, and not just for its local sections, which is an example of a possibly indefinite **bundle metric**.

Example

For any pseudo-Riemannian manifold (M, g) of dimension n we have

$$\mathrm{tr}_g(g) = g(g, g) \equiv n.$$

Next, we come to a process called raising, respectively lowering, indices of tensor fields. This is a most commonly used utility in theoretical physics, in particular general relativity.

Proposition A

Let (M, g) be a pseudo-Riemannian manifold. Then $T^{r,s}M \to M$ and $T^{r',s'}M \to M$ are isomorphic as vector bundles if $r + s = r' + s'$.

Proof:

- first show that $T^*M \to M$ and $TM \to M$ are **isomorphic**
- define $F : TM \to T^*M, \quad v \mapsto g(v, \cdot)$
- obtain $g(v, \cdot) \in T_p^*M$ for all $v \in T_pM$
- F is **smooth**, **fibre-preserving**, and **at each point a linear isomorphism**

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(continuation of proof)

- F^{-1} is given by

$$F^{-1}: T^*M \to TM, \quad \omega \mapsto g^{-1}(\omega, \cdot),$$

  where we use the **pointwise identification** $(T_p^*M)^* = T_pM$ and g^{-1} is given in local coordinates by

$$g^{-1} = \sum g^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}$$

- to show that $T^{r,s}M \to M$ and $T^{r',s'}M \to M$ are isomorphic for **arbitrary** r, s, r', s' with $r + s = r' + s'$ one **inductively** uses **entrywise isomorphisms**
- **note:** there are usually **choices** involved which vector or covector parts to change into covector and vector parts, respectively □

Remark

Proposition A is, in local coordinates, known as the process of **lowering and raising indices**. The isomorphisms of vector bundles $T^{r,s}M \to T^{r+1,s-1}M$ are denoted by $\sharp$ (read: **sharp**), and the isomorphisms $T^{r,s}M \to T^{r-1,s+1}M$ are denoted by $\flat$ (read: **flat**). Hence the name **musical isomorphisms**.

Our first application of the above is the generalization of the **gradient of a function** to pseudo-Riemannian manifolds:

Definition

Let (M, g) be a pseudo-Riemannian mfd. & $f \in C^\infty(M)$. The **gradient vector field of** f **w.r.t.** g, $\mathrm{grad}_g(f) \in \mathfrak{X}(M)$, is defined as

$$\mathrm{grad}_g(f) := g^{-1}(df) \in \mathfrak{X}(M).$$

In local coordinates $(x^1, \dots, x^n)$, $\mathrm{grad}_g(f) = \sum_{i,j=1}^{n} \frac{\partial f}{\partial x^i} g^{ij} \frac{\partial}{\partial x^j}$.

Gradient vector fields are important tools in the study of **pseudo-Riemannian submanifolds**:

Lemma A

Let $(\overline{M}, g)$ be a pseudo-Riemannian manifold, $M \subset \overline{M}$ a **pseudo-Riemannian submanifold of codimension** k, and identify $T_qM = \iota_*(T_qM) \subset T_q\overline{M}$ for all $q \in M$, where ι is the inclusion. For $p \in M$ fixed let $f = (f^1, \ldots, f^k) : U \to \mathbb{R}^k$, $U \subset \overline{M}$ open, $p \in U$, be any smooth map of **maximal rank** such that

$$M \cap U = \{f = 0\} \subset \overline{M}.$$

Then

$$T_qM = \ker(df_q^1) \cap \ldots \cap \ker(df_q^k) \subset T_q\overline{M}$$

and

$$(T_qM)^\perp = \mathrm{span}_{\mathbb{R}}\{\mathrm{grad}_g(f^1)_q, \ldots, \mathrm{grad}_g(f^k)_q\} \subset T_q\overline{M}$$

for all $q \in M \cap U$. In particular, $T_qM \oplus (T_qM)^\perp = T_q\overline{M}$ for all $q \in M \cap U$.

Proof: (next page)

(continuation of proof)

- **fix** $q \in M \cap U$ and $v \in T_q M$
- for any smooth curve $\gamma : I \to M \subset \overline{M}$, $\gamma'(t)$ is **tangential** to M $\forall t \in I$, follows by using **adapted coordinates**
- choose a smooth curve $\gamma : (-\varepsilon, \varepsilon) \to M \subset \overline{M}$ fulfilling $\gamma'(0) = v$
- $\rightsquigarrow \forall 1 \leq i \leq k$:

$$df^i(v) = \left.\frac{\partial}{\partial t}\right|_{t=0} (f \circ \gamma) = \left.\frac{\partial}{\partial t}\right|_{t=0} (0) = 0$$

- hence: $T_q M \subset \ker(df_q^1) \cap \ldots \cap \ker(df_q^k)$
- for other direction: f being of **maximal rank** implies $df_q^1, \ldots, df_q^k$ are **linearly independent**
- this implies

$$\dim(\ker(df_q^1) \cap \ldots \cap \ker(df_q^k)) = \dim(T_q \overline{M}) - k = \dim(T_q M).$$

- by **comparing dimensions**, obtain $T_q M \supset \ker(df_q^1) \cap \ldots \cap \ker(df_q^k)$ (continued on next page)

(continuation of proof)

- for $(T_qM)^\perp = \mathrm{span}_{\mathbb{R}}\{\mathrm{grad}_g(f^1)_q, \ldots, \mathrm{grad}_g(f^k)_q\}$ use that g is **pointwise nondegenerate**
- obtain that each **nonzero vector** in $\mathrm{span}_{\mathbb{R}}\{\mathrm{grad}_g(f^1)_q, \ldots, \mathrm{grad}_g(f^k)_q\}$ is **not contained** in $T_qM = \ker(df^1_q) \cap \ldots \cap \ker(df^k_q)$
- by $T_qM \oplus (T_qM)^\perp = T_q\overline{M}$ and **comparing dimensions**, the above claim follows □

Question: How can we make sense of $T_qM \oplus (T_qM)^\perp = T_q\overline{M}$ in a **coordinate free, global statement**?

Answer: Introduce **vector bundles along submanifolds** and **subbundles**!

Lemma B

Let $\pi_E : E \to \overline{M}$ be a vector bundle of rank k and M be a submanifold of $\overline{M}$. Then

$$\pi_{E|_M} : E|_M \to M, \quad (E|_M)_p := \pi_{E|_M}^{-1}(p) := \pi_E^{-1}(p) \quad \forall p \in M,$$
$$E|_M := \bigsqcup_{p \in M} (E|_M)_p,$$

is a vector bundle of rank k over M. It is called **vector bundle along** M.

Proof:

- suffices to work in **local coordinates**
- w.l.o.g. assume that locally, M is given by an **open set in** $\mathbb{R}^\ell$, $\ell \leq \dim(\overline{M})$, and the inclusion $M \subset \overline{M}$ is of the form $\iota : (x^1, \dots, x^\ell) \mapsto (x^1, \dots, x^\ell, 0, \dots, 0) \in \mathbb{R}^{\dim(\overline{M})}$
- next, apply **vector bundle chart lemma** to the restriction of, after possibly shrinking U, the **transition functions of** $E \to \overline{M}$ in local coordinates to $U \subset \mathbb{R}^{\dim(\overline{M})}$
- obtain that the **vector parts** are, still, **smooth** □

Heuristically, Lemma B means that we make the base manifold smaller but keep all possible vectors at each point in the vector bundle. **Combining** Lemma A & B, we get the following:

Definition

Let $(\overline{M}, g)$ be a pseudo-Riemannian manifold and $M \subset \overline{M}$ a pseudo-Riemannian submanifold of codimension k. Then the **normal bundle of** M, $TM^{\perp} \to M$, is defined as

$$TM^{\perp} := \bigsqcup_{p \in M} (T_p M)^{\perp},$$

with projection induced by the tangent bundle of $\overline{M}$ along M, $T\overline{M}|_M \to M$. In particular we have

$$T\overline{M}|_M = TM \oplus TM^{\perp},$$

and the above direct sum is **orthogonal with respect to** g.

The above definition can be put into a more general context, namely that of **subbundles**. (next page)

Definition

Let $\pi_E : E \to M$ be a vector bundle. Another vector bundle $\pi_F : F \to M$ is called **subbundle of $E \to M$** if for all $p \in M$, F_p is a **linear subspace** of E_p, the **canonical injection**

$$F \hookrightarrow E,$$

given fibrewise by the inclusion $F_p \subset E_p$, is an **embedding**, $\pi_F = \pi_E\big|_F$, and for **all local trivializations** ϕ of E **the restrictions** $\phi|_F$ **are local trivializations** of F. This means that the **bundle structure** of $F \to M$ and the **smooth manifold structure** of the total space F **are induced** by the bundle structure of $E \to M$ and the smooth manifold structure of the total space E, respectively.

Note: In the above sense, $TM \to M$ and $TM^\perp \to M$ of a pseudo-Riemannian submanifold $M \subset \overline{M}$ are both **subbundles of $T\overline{M}|_M \to M$**.

Examples

- Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$, $f(u^1, \dots, u^n) = \sum_i (u^i)^2$ and consider the ambient space $\mathbb{R}^{n+1}$ equipped with its standard Riemannian metric, denoted simply by $\langle \cdot, \cdot \rangle$. Then

$$S^n = \{f = 1\} \subset \mathbb{R}^{n+1}$$

is a **Riemannian submanifold** of $(\mathbb{R}^{n+1}, \langle \cdot, \cdot \rangle)$ with induced Riemannian metric

$$g := \langle \cdot, \cdot \rangle|_{TS^n \times TS^n}.$$

The **normal bundle** of $S^n \subset \mathbb{R}^{n+1}$, $TS^{n\perp}$, is spanned by the **position vector field** $\xi \in \mathfrak{X}(\mathbb{R}^{n+1})$ along S^n,

$$\xi : p \mapsto p \quad \forall p \in \mathbb{R}^{n+1}.$$

The **tangent bundle** of TS^n, viewed as a subbundle of $T\mathbb{R}^{n+1}|_{S^n}$, is thus fibrewise given by

$$T_pS^n = \ker(\langle \xi_p, \cdot \rangle) \subset T_p\mathbb{R}^{n+1}.$$

(continued on next page)

Examples (continuation)

- This means that a vector field X along S^n is **tangential** to S^n if and only if $\langle \xi, X\rangle \equiv 0$. Note that the function f used to define S^n fulfils $f = \langle \xi, \xi\rangle$.
- Next consider $\mathbb{R}^{n+1}$ but now equipped with a **pseudo-Riemannian metric** given in canonical coordinates by

$$\langle\cdot,\cdot\rangle_\nu := \sum_{i=1}^{n-\nu} du^i \otimes du^i - \sum_{i=n-\nu+1}^{n} du^i \otimes du^i .$$

Let $\xi \in \mathfrak{X}(\mathbb{R}^{n+1})$ denote the position vector field and define $f : \mathbb{R}^{n+1} \to \mathbb{R}$, $f := \langle \xi, \xi\rangle$. Then the level sets $\{f = -1\}$ are called **hyperboloids**,

$$H^n_\nu := \left\{ \langle \xi, \xi\rangle = \sum_{i=1}^{n-\nu+1} (u^i)^2 - \sum_{i=n-\nu+2}^{n+1} (u^i)^2 = -1 \right\} \subset \mathbb{R}^{n+1}.$$

Hyperboloids in $(\mathbb{R}^{n+1}, \langle\cdot,\cdot\rangle_\nu)$ are ***n*-dimensional ps.-R. manifolds** with induced metric of **index** $\nu - 1$.
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Examples (continuation)

- As for S^n,

$$T_pH^n_\nu = \ker(\langle \xi_p, \cdot \rangle_\nu) \subset T_p\mathbb{R}^{n+1}$$

and

$$T_pH^{n\,\perp}_\nu = \mathbb{R}\xi_p,$$

where $\mathbb{R}\xi_p$ is another commonly used notation for the linear span of **one** vector, that is $\mathrm{span}_{\mathbb{R}}\{\xi_p\}$. In the case $n = 3$, $\nu = 1$, H^3_1 is known as **two-sheeted hyperboloid**, and for $n = 3$, $\nu = 2$, H^3_2 is the **one-sheeted hyperboloid**.

END OF LECTURE 13

Next lecture:

- frames of vector bundles
- more subbundles
- Killing vector fields