Differential geometry
Lecture 12: Pseudo-Riemannian manifolds

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Recap of lecture 11:

- studied **tensor product** of vector bundles
- defined **tensor fields**, their possible **contractions**, pullback & pushforward
- discussed tensor fields as $C^\infty(M)$-**multilinear maps**
- defined **Lie derivative of tensor fields**, showed that it is a tensor derivation

Recall the following from linear algebra:

Remark

- a **pseudo-Euclidean scalar product** on a finite-dimensional real vector space V is a nondegenerate symmetric bilinear map

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$$

- nondegenerate $:= \nexists$ proper linear subspace $W \subset V$, such that $\langle \cdot, \cdot \rangle|_{W \times V} \equiv 0$
- V, together with $\langle \cdot, \cdot \rangle$ is called **pseudo-Euclidean vector space**
- the **index of** $\langle \cdot, \cdot \rangle$ is the number of its negative eigenvalues when viewed as symmetric $\dim(V) \times \dim(V)$-matrix
- the index of a pseudo-Euclidean scalar product is **basis-independent [**Sylvester's law of inertia**]**
- if the index vanishes, $(V, \langle \cdot, \cdot \rangle)$ is called **Euclidean vector space**

Examples

- $\mathbb{R}^n$ together with the Euclidean scalar product that is given by the dot-product

$$\langle v, w \rangle = \sum_{i=1}^{n} v^i w^i,$$

- $\mathbb{R}^{n+1}$ together with the **Minkowski scalar product**

$$\langle v, w \rangle = -v^{n+1} w^{n+1} + \sum_{i=1}^{n} v^i w^i.$$

Pseudo-Euclidean scalar product allow us to define the **length of vectors** and characterize vectors based on the **sign** of the scalar product with themselves. (next page)

Definition

The **length** of $v \in V$, $(V, \langle\cdot,\cdot\rangle)$ pseudo-Euclidean vector space, is defined as

$$\|v\| := \sqrt{|\langle v, v\rangle|}.$$

$\|\cdot\|$ is a **norm** on V if and only if $\langle\cdot,\cdot\rangle$ is Euclidean. One further says that a vector v is

- **spacelike** if $\langle v, v\rangle > 0$,
- **timelike** if $\langle v, v\rangle < 0$,
- **null** if $\langle v, v\rangle = 0$.

If $\langle\cdot,\cdot\rangle$ is **Euclidean**, each nonzero vector has **positive length**.

Recall: Two pseudo-Euclidean vector spaces $(V, \langle\cdot,\cdot\rangle_V)$ and $(W, \langle\cdot,\cdot\rangle_W)$ are called **isometric** if $\exists$ a linear isomorphism $A : V \to W$, such that $\langle\cdot,\cdot\rangle_V = \langle A\cdot, A\cdot\rangle_W$. A is then called **(linear) isometry**.

$\rightsquigarrow$ have the following **classification result**:

Proposition

Two finite-dimensional pseudo-Euclidean vector spaces are isometric if and only if their **dimension** and **index** of the scalar product **coincide**.

The above proposition means that any given pseudo-Euclidean vector space $(V, \langle \cdot, \cdot \rangle)$, $\dim(V) = n$, index of $\langle \cdot, \cdot \rangle = \nu$, is isometric to $(\mathbb{R}^n, \langle \cdot, \cdot \rangle_\nu)$, where

$$\langle v, v \rangle_\nu := \sum_{i=1}^{n-\nu} (v^i)^2 - \sum_{i=n-\nu+1}^{n} (v^i)^2.$$

Note: A pseudo-Euclidean scalar product might be interpreted as an element in $\mathrm{Sym}^2(V^*)$ which denotes the set of **symmetric two-tensors** in $V^* \otimes V^*$.

For our studies we need the concept of **orthogonality**.

Definition

Let $(V, \langle \cdot, \cdot \rangle)$ be a ps.-E. VS, $W \subset V$ a ps.-E. linear subspace, meaning that $\langle \cdot, \cdot \rangle|_{W \times W}$ is a pseudo-Euclidean scalar product on W. Then the **orthogonal complement** $W^\perp \subset V$ of W in V with respect to $\langle \cdot, \cdot \rangle$ is given by

$$W^\perp := \{v \in V \mid \langle v, w \rangle = 0 \quad \forall w \in W\}.$$

$W^\perp$ is a linear subspace of V of dimension $\dim(W^\perp) = \dim(V) - \dim(W)$ and

$$W \oplus W^\perp = V.$$

If $W \subset V$ is any linear subspace of V, we will also use the notation $W^\perp$ for its orthogonal complement. Two vectors $v, w \in V$ are called **orthogonal** if $\langle v, w \rangle = 0$, two linear subspaces V_1, V_2 of V are called **orthogonal** to each other if $\langle v_1, v_2 \rangle = 0$ for all $v_1 \in V_1$, $v_2 \in V_2$. A basis $\{v_1, \ldots, v_n\}$ of V is an **orthogonal basis** with respect to $\langle \cdot, \cdot \rangle$ if $\langle v_i, v_j \rangle = 0$ for all $i \neq j$, and **orthonormal basis** if additionally $\|v_i\| = 1$ for all $1 \leq i \leq n$.

Some facts:

- every pseudo-Euclidean vector space admits an orthonormal basis
- the index ν of a pseudo-Euclidean scalar product coincides with the number of elements in $\{i \mid \langle v_i, v_i\rangle = -1\}$ for any given orthonormal basis $\{v_1, \ldots, v_n\}$ of $(V, \langle\cdot,\cdot\rangle)$
- $(W^\perp)^\perp = W$ for all linear subspaces $W \subset V$
- W is a pseudo-Euclidean linear subspace $\Leftrightarrow$ $W \cap W^\perp = \{0\} \Leftrightarrow V = W \oplus W^\perp$
- linear isometries map orthonormal (orthogonal) bases to orthonormal (orthogonal) bases

Question: How do we, conceptually, go from pseudo-Euclidean vector spaces to smooth manifolds?
Answer: For each point p in a given manifold M define a pseudo-Euclidean scalar product on T_pM, such that this assignment **varies smoothly** on M!

Definition

A **pseudo-Riemannian metric with index** $0 \leq \nu \leq \dim(M)$ on a smooth mfd. M is a **symmetric** $(0,2)$-**tensor field** $g \in \mathcal{T}^{0,2}(M)$, $g : p \mapsto g_p \in \mathrm{Sym}^2(T_p^* M)$, such that for all $p \in M$ g_p is a **pseudo-Euclidean scalar product** of index ν on $T_p M$. This in particular means that

$$g(X, Y) = g(Y, X) \in C^\infty(M)$$

for all vector fields $X, Y \in \mathfrak{X}(M)$. If $\nu = 0$, g is called **Riemannian metric**. In local coordinates $(x^1, \dots, x^n)$ on $U \subset M$, g is of the form

$$g = \sum_{i,j=1}^{n} g_{ij} dx^i \otimes dx^j,$$

where

$$g_{ij} := g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \in C^\infty(U) \quad \forall 1 \leq i, j \leq n.$$

Remark

- The **symmetry condition** for g is equivalent to requiring that in all local coordinates $g_{ij} = g_{ji}$. This means that (g_{ij}), **viewed as a $n \times n$-matrix valued smooth map** on the coordinate domain, is at each point a **symmetric matrix**.
- If we write in **local coordinates** $X = \sum_{i=1}^{n} X^i \frac{\partial}{\partial x^i}$, $Y = \sum_{i=1}^{n} Y^i \frac{\partial}{\partial x^i}$, we obtain the **local formula** for $g(X, Y)$

$$g(X, Y) = \sum_{i,j=1}^{n} g_{ij} X^i Y^j.$$

- **Heuristically:** Plug in X in the left half and Y in the right half of the tensor terms in g.

Now we can finally define the objects of **main interest** of this course:

Definition

A smooth manifold M equipped with a (pseudo)-Riemannian metric g is called **(pseudo)-Riemannian manifold**.

An immediate use of a Riemannian metric is:

Definition

Let (M, g) be a Riemannian manifold and $\gamma : I \to M$ a smooth curve. Then the **arc-length**, or simply **length**, of γ is defined as

$$L(\gamma) = \int_I \sqrt{g(\gamma', \gamma')} dt.$$

Note that $L(\gamma) = \infty$ is allowed.

Let us take a look at examples of pseudo-Riemannian manifolds:

Examples

- Any **pseudo-Euclidean vector space** $(V, \langle\cdot,\cdot\rangle)$ is, viewed as a smooth manifold with $g_p := \langle\cdot,\cdot\rangle$ for all $p \in V$. If $V = \mathbb{R}^n$ equipped with its canonical coordinates and Euclidean scalar product at **each** tangent space, the induced Riemannian metric in canonical coordinates $(u^1, \dots, u^n)$ is given by

$$g = \sum_{i=1}^{n} du^i \otimes du^i .$$

- Any **smooth submanifold** $M \subset \mathbb{R}^n$ equipped with

$$g \in \mathcal{T}^{0,2}(M), \quad g_p = \langle\cdot,\cdot\rangle|_{T_pM \times T_pM},$$

for all $p \in M$, that is the **restriction of the Euclidean scalar product** at origin $p \in \mathbb{R}^n$ to the tangent space of M at p.

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Examples (continuation)

- More generally, any **smooth submanifold of a smooth Riemannian manifold** is by restriction of the metric to the tangent bundle of the smooth submanifold a Riemannian manifold.
- If (M, g_M) and (N, g_N) are pseudo-Riemannian manifolds and g_M, g_N, have index ν_M, ν_N, respectively, the **product** $M \times N$ is a pseudo-Riemannian manifold of index $\nu_M + \nu_N$. The metric on $M \times N$ is given by

$$g_{M\times N} := g_M + g_N,$$
$$g_{M\times N}((v,w),(v,w)) = g_M(v,v) + g_N(w,w),$$

for all $(v,w) \in TM \oplus TN \cong T(M \times N)$. The metric $g_{M\times N}$ is called **product metric**.

When studying submanifolds of pseudo-Riemannian manifolds with index $1 \leq \nu < \dim(M)$, one has to be very careful as the restriction of the metric **might not be a pseudo-Riemannian metric** on the submanifold, e.g. the diagonal line in the 2-dim. Minkowski space. However, we have the following definition:

Definition

Let $(N, \overline{g})$ be a **pseudo-Riemannian manifold** and $M \subset N$ a **smooth submanifold**. M is called **pseudo-Riemannian submanifold** of N if

$$g := \overline{g}|_{TM \times TM}$$

is a pseudo-Riemannian metric on M.

Note: Restricting g to $TM \times TM$ means that we restrict the **basepoint** of $\overline{g}$ to $M \subset N$ and the **vectors** we are allowed to plug in to vectors in $TM \subset TN$.

While not every manifold admits a pseudo-Riemannian metric for any given index, we have the following existence result for **Riemannian metrics**:

Proposition

Let M be a smooth manifold. Then there **exists** a Riemannian metric g on M.

Proof:

- choose countable atlas $\{(\varphi_i, U_i) \mid i \in I\}$ of M and countable locally finite subordinate partition of unity $\{b_i,\ i \in I\}$ of M
- define $g := \sum_{i \in I} b_i \langle d\varphi \cdot, d\varphi \cdot \rangle = \sum_{i \in I} b_i \varphi^* \langle \cdot, \cdot \rangle$, where $\langle \cdot, \cdot \rangle$ denotes standard Riemannian metric on $\mathbb{R}^{\dim(M)}$
- check that g well-defined since sum locally finite and at each point positive definite $\square$

Note: g is **far** from unique!

We know what **diffeomorphic** means for manifolds, and **isometric** for pseudo-Euclidean vector spaces. For pseudo-Riemannian manifolds, the two definitions are combined:

Definition

Let (M, g) and (N, h) be pseudo-Riemannian manifolds and $F : M \to N$ a **diffeomorphism**. Then F is called an **isometry** if $F^* h = g$ or, equivalently, $F_* g = h$. One checks that the first condition is equivalent to

$$g_p(X_p, Y_p) = h_{F(p)}(dF_p(X_p), dF_p(Y_p))$$

for all $X, Y \in \mathfrak{X}(M)$ and all $p \in M$, meaning that pointwise dF_p is a **linear isometry**. The two pseudo-Riemannian manifolds (M, g) and (N, h) are then called **isometric**.

Note: The isometries $F : M \to M$ for (M, g) form a group, **the isometry group of** (M, g), which is denoted by $\mathrm{Isom}(M, g)$.

Examples

- Every **orthogonal transformation** $A \in \mathrm{O}(n+1)$ is, by definition, an isometry of $\mathbb{R}^{n+1}$ equipped with the standard Riemannian metric given pointwise by the Euclidean scalar product $\langle \cdot, \cdot \rangle$.
- Since each $A \in \mathrm{O}(n+1)$ **restricts** to a diffeomorphism of $S^n \subset \mathbb{R}^{n+1}$, it is an isometry of $(S^n, \langle \cdot, \cdot \rangle|_{TS^n \times TS^n})$. The Riemannian metric $\langle \cdot, \cdot \rangle|_{TS^n \times TS^n}$ is sometimes called the **round metric**.
- The upper half plane $H := \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ equipped with the Riemannian **Poincaré metric**

$$g = \frac{1}{y^2}(dx^2 + dy^2),$$

  is called the **Poincaré half-plane model**.

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Examples (continuation)

When viewed as a subset of $\mathbb{C}$ via $H \ni (x, y) \mapsto x + iy \in \mathbb{C}$, one obtains an **isometric action** of

$$\mathrm{PSL}(2, \mathbb{R}) = \mathrm{SL}(2, \mathbb{R})/_{\sim}, \quad A \sim B :\Leftrightarrow A = \pm B$$

on $H \subset \mathbb{C}$ defined by

$$\mu : \mathrm{PSL}(2, \mathbb{R}) \times H \to H, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z := \frac{az + b}{cz + d}.$$

coordinate change in $M \rightsquigarrow$ pointwise **change of basis** in TM
We obtain the following transformation rule for ps.-R. metrics:

Lemma

Let (M, g) be a pseudo-Riemannian manifold and $\varphi = (x^1, \dots, x^n)$, $\psi = (y^1, \dots, y^n)$, be **local coordinate systems** on $U \subset M$, respectively $V \subset M$, such that $U \cap V \neq \emptyset$. Denote on $U \cap V$

$$g = \sum_{i,j} g_{ij} dx^i \otimes dx^j = \sum_{i,j} \widetilde{g}_{ij} dy^i \otimes dy^j.$$

φ and ψ are related by $(x^1, \dots, x^n) = F(y^1, \dots, y^n)$ on $U \cap V$, where $F : \psi(U \cap V) \to \varphi(U \cap V)$.Then the **matrix valued maps** (g_{ij}) and $(\widetilde{g}_{ij})$ in the above equation are related by

$$(\widetilde{g}_{ij})|_p = dF^T_{\psi(p)} \cdot (g_{ij})|_{\varphi^{-1}(F(\psi(p)))} \cdot dF_{\psi(p)}.$$

Proof: Follows by considering **coordinate representations** of (g_{ij}) and $(\widetilde{g}_{ij})$, writing down the pullback of (g_{ij}) with respect to F, and comparing the prefactors. □

END OF LECTURE 12

Next lecture:

- trace with respect to a pseudo-Riemannian metric
- induced tensor bundle metric
- raising/lowering indices
- subbundles, in particular tangent bundles of submanifolds