Differential geometry
Lecture 11: Tensor bundles and tensor fields

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Recap of lecture 10:

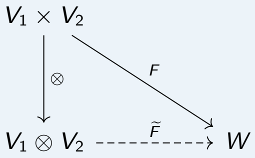
- defined **dual vector bundle** $E^* \to M$ for a given vector bundle $E \to M$, in particular the **cotangent bundle** $T^*M \to M$
- studied **1-forms** $\Omega^1(M)$, that is sections in $T^*M \to M$, interpreted them as "dual" to vector fields
- defined the **direct sum** of vector bundles, called the **Whitney sum**

Last lecture, we described how to construct from the pointwise dual and the pointwise direct sum of fibres of vector bundles the dual vector bundle and the Whitney sum, respectively.

Next construction: the **tensor product** of vector bundles.

Remark

Recall that the **tensor product of two real vector spaces** V_1, $\dim(V_1) = n$, and V_2, $\dim(V_2) = m$, is a real vector space $V_1 \otimes V_2$ **determined up to linear isomorphy** together with a bilinear map $\otimes : V_1 \times V_2 \to V_1 \otimes V_2$, such that for every real vector space W and every bilinear map $F : V_1 \times V_2 \to W$, there exist a **unique** linear map $\widetilde{F} : V_1 \otimes V_2 \to W$ making the diagram



commute. This is the **defining universal property** of the tensor product. (continued on next page)

Remark (continuation)

The **dimension** of $V_1 \otimes V_2$ is $n \cdot m$. If $\{v_1^1, \dots, v_1^n\}$ and $\{v_2^1, \dots, v_2^m\}$ are a basis of V_1 and V_2, respectively, we can construct a choice of basis for $V_1 \otimes V_2$ **explicitly**. A **basis of** $V_1 \otimes V_2$ is given by $\{v_1^i \otimes v_2^j,\ 1 \leq i \leq n,\ 1 \leq j \leq m\}$, and $\widetilde{F}$ for a bilinear map F as on the previous slide given by

$$\widetilde{F} : v_1^i \otimes v_2^j \mapsto F(v_1^i, v_2^j)$$

on the basis vectors. By considering "$\otimes$" itself as a bilinear map from $V_1 \times V_2$ to $W = V_1 \otimes V_2$, we define $v \otimes w$ for $v = \sum_{i=1}^{n} a_i v_1^i$, $w = \sum_{j=1}^{m} b_j v_2^j$, as

$$v \otimes w := \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j \cdot v_1^i \otimes v_2^j.$$

The above equation is **consistent** with the definition of $\widetilde{F}$. An element $v \in V_1 \otimes V_2$ is called a **pure tensor** if it can be written as $v = v_1 \otimes v_2$ for some $v_1 \in V_1$ and $v_2 \in V_2$.

Observe that $V \otimes \mathbb{R} \cong V \cong \mathbb{R} \otimes V$. We can, in this special case, interpret "$\otimes$" as **scalar multiplication**. Before coming to tensor products of vector bundles, we need to be aware of the following facts:

Remark

- The real **vector space of endomorphisms** $\mathrm{End}(V)$ and $V \otimes V^*$ are isomorphic as real vector spaces via

$$V \otimes V^* \ni v \otimes \omega \mapsto (u \mapsto \omega(u)v) \in \mathrm{End}(V).$$

- For the **evaluation map**

$$\mathrm{ev} : V \times V^* \to \mathbb{R}, \quad (v, \omega) \mapsto \omega(v) \quad \forall v \in V,\ \omega \in V^*,$$

  the induced map $\widetilde{\mathrm{ev}} : V \otimes V^* \to \mathbb{R}$ is called **contraction**. By saying that we **contract** $v \otimes \omega$ we simply mean sending it to $\omega(v)$.
- $V_1 \otimes (V_2 \otimes V_3)$ and $(V_1 \otimes V_2) \otimes V_3$ are **isomorphic**.
- $V_1 \otimes V_2$ and $V_2 \otimes V_1$ are **isomorphic**.

Remark (continuation)

- For tensor products of vector bundles we will **pointwise** deal with objects of the form

$$V^{\otimes r} \otimes V^{*\otimes s} := \underbrace{V \otimes \ldots \otimes V}_{r \text{ times}} \otimes \underbrace{V^* \otimes \ldots \otimes V^*}_{s \text{ times}}.$$

A **contraction** of an element
$v_1 \otimes \ldots \otimes v_r \otimes \omega_1 \otimes \ldots \otimes \omega_s \in V \otimes \ldots \otimes V \otimes V^* \otimes \ldots \otimes V^*$
will stand for a map of the form

$$v_1 \otimes \ldots \otimes v_r \otimes \omega_1 \otimes \ldots \otimes \omega_s \mapsto$$

$$\omega_\beta(v_\alpha) \cdot v_1 \otimes \ldots \widehat{\otimes\ v_\alpha} \otimes \ldots v_r \otimes \omega_1 \otimes \ldots \widehat{\otimes\ \omega_\beta} \otimes \ldots \omega_s$$

for $1 \leq \alpha \leq r$ and $1 \leq \beta \leq s$ fixed, where "$\widehat{\ }$" means that the element is supposed to be **left out**. This is precisely the **induced map** for the **evaluation map in the (α, β)-th entry**.

Now that we have refreshed our knowledge of the tensor product we can define the tensor product of vector bundles:

Definition

Let $\pi_E : E \to M$ be a vector bundle of rank k and $\pi_F : F \to M$ be a vector bundle of rank ℓ and, as for the Whitney sum, let ψ_i^E and ψ_i^F, $i \in I$, be local trivializations of E and F, respectively, and $\mathcal{A}$ a fitting atlas of M with charts (φ_i, U_i), $i \in I$. The **tensor product of vector bundles** of E and F, $\pi_{E\otimes F} : E \otimes F \to M$, is the vector bundle given pointwise by

$$(E \otimes F)_p = \pi_{E\otimes F}^{-1}(p) := E_p \otimes F_p,$$

so that $E \otimes F := \bigsqcup_{p\in M} E_p \otimes F_p$.

Remark: By the vector bundle chart lemma, it suffices to construct local trivializations $\phi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^k \otimes \mathbb{R}^\ell \cong U_i \times \mathbb{R}^{k\ell}$ covering $E \otimes F$ with smooth vector parts of their transition functions in order to show that $E \otimes F$ **is in fact a vector bundle**. (continued on next page)

(continuation of remark)

- analogous to construction of **Whitney sum** we set

$$\phi_i^{-1} := (\psi_i^E \otimes \psi_i^F)^{-1} \circ (\Delta_M \times \mathrm{id}_{\mathbb{R}^{k\ell}}) :$$
$$U_i \times \mathbb{R}^{k\ell} \cong U_i \times (\mathbb{R}^k \otimes \mathbb{R}^\ell) \to \bigsqcup_{p \in U_i} (E_p \otimes F_p),$$
$$(p, v \otimes w) \mapsto (\psi_i^E)^{-1}(p, v) \otimes (\psi_i^F)^{-1}(p, w)$$
$$\forall p \in U_i,\ v \in \mathbb{R}^k,\ w \in \mathbb{R}^\ell,$$

  where $\Delta_M : p \mapsto (p, p) \in M \times M$ again denotes the diagonal embedding and ϕ_i^{-1} on **non-pure tensors** is defined by **linear extension** for any $p \in U_i$ fixed

- for the transition functions of the **vector part** in the change of local trivializations of $E \otimes F \to M$ we obtain for all $i, j \in I$, such that $U_i \cap U_j \neq \emptyset$,

$$\phi_i \circ \phi_j^{-1}(p, v \otimes w) = (p, \tau_{ij}^E(p) v \otimes \tau_{ij}^F(p) w),$$

  where τ_{ij}^E and τ_{ij}^F are the **transition functions of the local trivializations of E and F**, respectively

(continuation of remark)

- observe: the linear extension of

$$\mathbb{R}^k \otimes \mathbb{R}^\ell \ni v \otimes w \mapsto \tau_{ij}^E(p)v \otimes \tau_{ij}^F(p)w \in \mathbb{R}^k \otimes \mathbb{R}^\ell$$

  is an invertible linear map and conclude with vector bundle chart lemma that $E \otimes F \to M$ **is indeed a vector bundle of rank** $k\ell$

Example

The **endomorphism bundle of a vector bundle** $E \to M$ is given by

$$\mathrm{End}(E) := E \otimes E^* \to M.$$

The transition functions of $\mathrm{End}(E) \to M$ induced by given transition functions τ_{ij} in the vector part on $E \to M$ are, in induced coordinates, of the form

$$(p, A) \mapsto (p, \tau_{ij}(p) \cdot A \cdot \tau_{ij}(p)^{-1}).$$

Having dealt with all technical necessities we can now define tensor fields.

Definition

Let M be a smooth manifold and let $(r,s) \in \mathbb{N}_0 \times \mathbb{N}_0$ so that $r+s>0$ **[** for now **]**. The vector bundle

$$T^{r,s}M := \underbrace{TM \otimes \ldots \otimes TM}_{r \text{ times}} \otimes \underbrace{T^*M \otimes \ldots \otimes T^*M}_{s \text{ times}} \to M$$

is called the **bundle of** (r,s)**-tensors** of M. In this notation, $T^{1,0}M = TM$ and $T^{0,1}M = T^*M$. The (local) sections in the bundle of (r,s)-tensors are called **(local)** (r,s)**-tensor fields**, or simply **tensor fields** if (r,s) is clear from the context, and are denoted by

$$\mathcal{T}^{r,s}(M) := \Gamma(T^{r,s}M).$$

Question: How do tensor fields look locally?
Answer:

Remark

In **local coordinates** $(x^1, \dots, x^n)$ on $U \subset M$, tensor fields $A \in \mathcal{T}^{r,s}(M)$ are of the form

$$A = \sum_{\substack{1 \le i_1, \dots, i_r \le n \\ 1 \le j_1, \dots, j_r \le n}} A^{i_1 \dots i_r}{}_{j_1 \dots j_s} \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_s},$$

$$A^{i_1 \dots i_r}{}_{j_1 \dots j_s} \in C^\infty(U) \quad \forall 1 \le i_1, \dots, i_r, j_1, \dots, j_s \le n.$$

The above local form of tensor fields is commonly called **index notation of tensor fields**. This is justified by the fact that locally A is **uniquely determined** by the local smooth functions $A^{i_1 \dots i_r}{}_{j_1 \dots j_s}$ on chart neighbourhoods of an atlas of M.

Recall that pointwise, we can contract elements in vector spaces of the form $V \otimes \ldots \otimes V \otimes V^* \otimes \ldots \otimes V^*$.

Question: What happens if we contract tensor fields at each point?

Answer:

Remark

If $A \in \mathcal{T}^{r,s}(M)$ with $r > 0$ and $s > 0$ we can **contract** A in the i, j-th index, $1 \leq i \leq r$, $1 \leq j \leq s$, which is **pointwise in local coordinates** defined as for contractions in $T_pM \otimes \ldots \otimes T_pM \otimes T_p^*M \otimes \ldots \otimes T_p^*M$, and **obtain an** $(r-1, s-1)$**-tensor field** $\widetilde{A} \in \mathcal{T}^{r-1,s-1}(M)$.

Problem: What if $r = s = 1$?

Solution: We define

$$\mathcal{T}^{0,0}(M) := C^\infty(M).$$

Question: Is this a good idea?

Answer: Yes! Since: (see next page)

Remark

Observe that for any $a \in \mathcal{T}^{r,s}(M)$, $b \in \mathcal{T}^{R,0}(M)$, $c \in \mathcal{T}^{0,S}(M)$,

$$b \otimes a \in \mathcal{T}^{r+R,s}(M), \quad a \otimes c \in \mathcal{T}^{r,s+S}(M),$$

where the tensor product is understood over $C^\infty(M)$ **[** means: pointwise over $\mathbb{R}$ **]**. This is **compatible** with $\mathcal{T}^{0,0}(M) = C^\infty(M)$ since $C^\infty(M) \otimes_{C^\infty(M)} \mathcal{T}^{r,s}(M) \cong \mathcal{T}^{r,s}(M)$, which pointwise corresponds to $\mathbb{R} \otimes T^{r,s}(M) \cong T^{r,s}(M)$, and the same with tensors from the left. **In practice** this just means

$$f \otimes a = a \otimes f := fa \quad \forall f \in C^\infty(M),\ a \in \mathcal{T}^{r,s}(M),$$

where the multiplication in fa is understood pointwise.

We know that for any real finite dimensional vector space V, $V^{\otimes r} \otimes V^{*\otimes s}$ is as a vector space isomorphic to the vector space of **multilinear maps** $\mathrm{Hom}_{\mathbb{R}}(V^{*\times r} \times V^{\times s}, \mathbb{R})$.
Question: How does this translate to tensor fields?
Answer:

Proposition

$\mathcal{T}^{r,s}(M)$ is as $C^\infty(M)$-module isomorphic to the $C^\infty(M)$-**multilinear maps**

$$\mathrm{Hom}_{C^\infty(M)}(\Omega^1(M)^{\times r} \times \mathfrak{X}(M)^{\times s}, C^\infty(M)).$$

Heuristically: If we are given a tensor field $A \in \mathcal{T}^{r,s}(M)$ we can "plug in" s **vector fields from the left** and r **1-forms from the right** and **obtain a smooth function on** M. In the special case that A is an **endomorphism field** of TM, these operations are in coordinate representations just **multiplication of a square matrix valued function**, a.k.a. A, with a **column vector valued function from the right**, a.k.a. a vector field X, with a **row vector valued function from the left**, a.k.a. a 1-form ω.

Next, just as for vector fields and 1-forms, we define:

Definition

Let M, N be smooth manifolds and let $F : M \to N$ be a diffeomorphism. The **pushforward and pullback of tensor fields** under F are the unique $\mathbb{R}$-linear maps

$$F_* : \mathcal{T}^{r,s}(M) \to \mathcal{T}^{r,s}(N), \quad F^* : \mathcal{T}^{r,s}(N) \to \mathcal{T}^{r,s}(M),$$

such that

- $F_* : \mathcal{T}^{1,0}(M) \to \mathcal{T}^{1,0}(N)$ is the **pushforward of vector fields**, $F^* : \mathcal{T}^{1,0}(N) \to \mathcal{T}^{1,0}(M)$ is the **pullback of vector fields**,
- $F_* : \mathcal{T}^{0,1}(M) \to \mathcal{T}^{0,1}(N)$ is the **pushforward of 1-forms**, $F^* : \mathcal{T}^{0,1}(N) \to \mathcal{T}^{0,1}(M)$ is the **pullback of 1-forms**,
- $F_*(b \otimes a) = (F_* b) \otimes (F_* a)$ and $F^*(b \otimes a) = (F^* b) \otimes (F^* a)$ for all $a \in \mathcal{T}^{r,s}(M)$, $b \in \mathcal{T}^{R,0}(M)$,
- $F_*(a \otimes c) = (F_* a) \otimes (F_* c)$ and $F^*(a \otimes c) = (F^* a) \otimes (F^* c)$ for all $a \in \mathcal{T}^{r,s}(M)$, $c \in \mathcal{T}^{0,S}(M)$.

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Definition (continuation)

For $f \in C^\infty(M)$, $g \in C^\infty(N)$, we set

$$F_*(f) := f \circ F^{-1}, \quad F^* g := g \circ F$$

so that $F_*(fa) = F_*(f)F_*(a)$ and $F^*(gb) = F^*(g)F^*(b)$ for all $f \in C^\infty(M)$, $g \in C^\infty(N)$, $a \in \mathcal{T}^{r,s}(M)$, $b \in \mathcal{T}^{r,s}(N)$.

Note: The above definition **looks worse than it actually is**. Locally, (r, s)-tensor fields are summations of smooth functions times tensor fields of the form

$$\frac{\partial}{\partial x^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_s}$$

Hence, for, say, the pullback of the above expression under some smooth map we only need to calculate the pullback of all possible **coordinate vector fields** $\frac{\partial}{\partial x^i}$ and **coordinate one forms** dx^j and then use the $C^\infty(M)$-**linearity** of the tensor product, e.g.

$$\frac{\partial}{\partial x} \otimes (f dx + g dy) = f \frac{\partial}{\partial x} \otimes dx + g \frac{\partial}{\partial x} \otimes dy.$$

Remark

Just like for 1-forms, the pullback of $(0, r)$-tensor fields ("pointwise only **covectors** tensored together") under a smooth map F is defined **regardless of whether F is a diffeomorphism or not**.

The pushforward and pullback of tensor fields has the following important property that justifies calculating **without coordinates** whenever possible:

Lemma

Contractions of tensor fields **commute** with the pushforward and with the pullback defined above.

Proof: It suffices to prove this statement for **endomorphism fields** which have only **one** possible contraction.
[Details: Exercise!**]** □

Lastly, we will study the **Lie derivative of tensor fields**, which is defined analogously to the Lie derivative of vector fields:

Definition

Let M be a smooth manifold, $X \in \mathfrak{X}(M)$ a vector field, and $A \in \mathcal{T}^{r,s}(M)$ a tensor field. Then the **Lie derivative of A in direction of X**, $\mathcal{L}_X A \in \mathcal{T}^{r,s}(M)$, is defined as

$$(\mathcal{L}_X A)_p := \frac{\partial}{\partial t}\bigg|_{t=0} (\varphi_t^* A)_p \quad \forall p \in M,$$

where $\varphi : I \times U \to M$ is any local flow of X near $p \in M$.

Question 1: What is the Lie derivative of $A \in \mathcal{T}^{0,0}(M) = C^\infty(M)$?
Question 2: Is the above definition as tedious to work with as it looks to be?
Answer 1: $\mathcal{L}_X A = X(A)$, meaning that for smooth functions the Lie derivative is simply the action of the vector field.
Answer 2: No! Because: (next page)

Proposition

The Lie derivative of tensor fields is a **tensor derivation**, i.e. it is **compatible with all possible contractions** and fulfils the **Leibniz rule**

$$\mathcal{L}_X(A \otimes B) = \mathcal{L}_X A \otimes B + A \otimes \mathcal{L}_X B$$

for all vector fields X and all tensor fields A, B, such that $A \otimes B$ is defined.

Proof:

- it suffices to prove this proposition for **endomorphism fields** $A \in \mathcal{T}^{1,1}(M)$ as all other possible cases will follow **by induction and the Leibniz rule**
- $\rightsquigarrow$ first need to prove that $\mathcal{L}_X$ **fulfils** Leibniz rule
- fix $p \in M$ & A, B tensor fields, such that $A \otimes B$ is defined
- first assume that $(A \otimes B)_p = A_p \otimes B_p \neq 0$, and let $X \in \mathfrak{X}(M)$ be arbitrary, denote by $\varphi : I \times U \to M$ its **local flow near** p with $U \subset M$ contained in a **chart neighbourhood** for some local coordinates on M

(continuation of proof)

- choose interval $(-\varepsilon, \varepsilon) \subset I$ for $\varepsilon > 0$ small enough, such that in the local coordinates on U and the **induced coordinates** on the fitting (r, s)-tensor bundles ψ and ϕ, the **pullbacks** of A and B w.r.t. the local flow of X are of the form

$$\psi((\varphi_t^* A)_p) = (p, a(t)v), \quad \phi((\varphi_t^* B)_p) = (p, b(t)w)$$

  $\forall t \in (-\varepsilon, \varepsilon)$
- in the above equation, $0 \neq v \in \mathbb{R}^{N_1}$ and $0 \neq w \in \mathbb{R}^{N_2}$ are **fixed nonzero vectors** and N_1, N_2, depend on the type of tensor field that A and B are
- the expressions $a(t)$ and $b(t)$ stand for **smooth and uniquely defined maps**

$$a : (-\varepsilon, \varepsilon) \to \mathrm{GL}(N_1), \quad b : (-\varepsilon, \varepsilon) \to \mathrm{GL}(N_2),$$

  with $a(0) = \mathrm{id}_{\mathbb{R}^{N_1}}$ and $b(0) = \mathrm{id}_{\mathbb{R}^{N_2}}$

(continuation of proof)

- $\rightsquigarrow$ in order to prove that the Leibniz is fulfilled, it suffices to show that for **any finite dimensional real vector spaces** V, $\dim(V) = N_1$, and W, $\dim(W) = N_2$, and **any smooth maps** a **and** b **as above**,

$$\frac{\partial}{\partial t}\bigg|_{t=0} ((a(t)v) \otimes (b(t)w)) = (a'(0)v) \otimes w + v \otimes (b'(0)) \tag{1}$$

  for all $v \in V$, $w \in W$

- follows from the **defining universal property of the tensor product of vector spaces**:
- let $L : V \times W \to \mathbb{R}$ be any **bilinear map** and $\widetilde{L} : V \otimes W \to \mathbb{R}$ the corresponding **linear map**, so that $L(a(t)v, b(t)w) = \widetilde{L}((a(t)v) \otimes (b(t)w))$ for all $v \in V$, $w \in W$, $t \in (-\varepsilon, \epsilon)$
- by taking the t-**derivative** at $t = 0$ **on both sides** we obtain

$$\frac{\partial}{\partial t}\bigg|_{t=0} \widetilde{L}((a(t)v) \otimes (b(t)w)) = \widetilde{L}((a'(0)v) \otimes w + v \otimes (b'(0)w))$$

(continuation of proof)

- since L and thus $\widetilde{L}$ were arbitrary, the above statement hold in particular for all **component functions**
- this shows equation (1) and, hence, **proves the Leibniz rule for** $\mathcal{L}_X$
- for **compatibility with contractions** it is enough to consider $V = W^*$ and $L = \mathrm{ev}$ the **evaluation map**
- $\rightsquigarrow$ $\widetilde{L}$ is precisely the contraction $\checkmark$
- now assume $(A \otimes B)_p = 0$ and that there exists a **convergent sequence** $\{p_n\}_{n\in\mathbb{N}}$ with $p_n \to p$ as $n \to \infty$, such that $(A \otimes B)_{p_n} \neq 0$ for all $n \in N$
- $\rightsquigarrow$ the statement of this proposition follows with a continuity argument similar to the one used in Proposition B, Lecture 9 **[**which is this proposition for $(r, s) = (1, 0)$**]**
- lastly assume $(A \otimes B)_p = 0$ and $A \otimes B$ **vanishes identically** on an open neighbourhood $U \subset M$ of p
- $\rightsquigarrow$ A **or** B must already **vanish identically** on U

(continuation of proof)

- w.l.o.g. assume that U is a **chart neighbourhood**, choose a fitting **bump function** b with $\mathrm{supp}(b) \subset U$ compactly embedded, so that the locally defined prefactors in the local forms of A and B, multiplied with b, are **globally defined smooth functions**
- now use that bA or bB **vanish identically** and in some smaller open neighbourhood $V \subset U$ **coincide with** A **and** B, respectively
- on V we obtain if $bA \equiv 0$ $\mathcal{L}_X(A) = \mathcal{L}_X(bA) = \mathcal{L}_X(0) = 0$ and a similar identity for B and $A \otimes B$ □

The result of the latter proposition tells us how to **actually calculate** $\mathcal{L}_X A$ for given $X \in \mathfrak{X}(M)$, $A \in \mathcal{T}^{r,s}(M)$. All that remains is to understand how the **Lie derivative of 1-forms** looks like:

Corollary

$(\mathcal{L}_X\alpha)(Y) = X(\alpha(Y)) - \alpha([X, Y])$ for all $X, Y \in \mathfrak{X}(M)$ and all $\alpha \in \Omega^1(M)$.

Proof: Follows from compatibility with contractions, that is $X(\alpha(Y)) = \mathcal{L}_X(\alpha(Y)) = (\mathcal{L}_X\alpha)(Y) + \alpha(\mathcal{L}_X Y)$. □

END OF LECTURE 11

Next lecture:

- pseudo-Euclidean vector spaces
- pseudo-Riemannian metrics