Differential geometry
Lecture 10: Dual bundles, 1-forms, and the Whitney sum

David Lindemann

University of Hamburg
Department of Mathematics
Analysis and Differential Geometry & RTG 1670

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Recap of lecture 9:

- defined **pushforward & pullback** of vector fields
- locally **rectified** vector fields
- proved that $[X, Y]$ measures **infinitesimal change** of Y along local flow of X
- defined **Lie derivative of vector fields**
- erratum: mixed up terms **one parameter families** and **one parameter groups**

Recall the definition of **dual vector spaces** from linear algebra: A real (for our purposes **finite dimensional**) vector space V has dual vector space $V^* := \{\omega : V \to \mathbb{R} \mid \omega \ \mathbb{R}\text{-linear map}\}$.
Question: How can we translate this concept to vector bundles?
Answer: Use the vector bundle chart lemma!

Definition

Let $\pi_E : E \to M$ be a vector bundle of rank k. The **dual vector bundle** $\pi_{E^*} : E^* \to M$ is pointwise given by

$$\pi_{E^*}^{-1}(p) = E_p^* := \mathrm{Hom}_{\mathbb{R}}(E_p, \mathbb{R})$$

for all $p \in M$.

The **topology**, **smooth manifold structure**, and **bundle structure** on E^* is obtained as follows:

- let $\{(\psi_i, V_i) \mid i \in A\}$ be a collection of **local trivializations** of a vector bundle E of rank k, such that there exists an atlas $\mathcal{A} = \{(\varphi_i, \pi_E(V_i)) \mid i \in A\}$ of M
- note: $\{V_i \mid i \in A\}$ is an **open covering** of E

(continuation)

- then $\mathcal{B} := \{((\varphi_i \times \mathrm{id}_{\mathbb{R}^k}) \circ \psi_i, V_i) \mid i \in A\}$ is an **atlas** on E
- recall: for any finite dimensional real vector space W, $(W^*)^*$ and W are **isomorphic** via

$$W \ni v \mapsto (\omega \mapsto \omega(v)),\ \omega \in W^*.$$

- the topology on E^* is given by pre-images of open images of the **dual local trivializations** which are defined by

$$\widetilde{\psi}_i : \pi_{E^*}^{-1}(\pi_E(V_i)) \to \pi_E(V_i) \times \mathbb{R}^k, \quad \omega_p \mapsto (p, w),$$

  where $w \in \mathbb{R}^k$ is the unique vector, such that $\omega_p(v_p) = \langle w, \mathrm{pr}_{\mathbb{R}^k}(\pi_E(v_p))\rangle$ for all $v_p \in \pi_E^{-1}(p)$ and $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product on $\mathbb{R}^k$ induced by its canonical coordinates
- the **dual atlas** $\mathcal{B}^*$ on E^* is then defined by

$$\mathcal{B}^* := \{((\varphi_i \times \mathrm{id}_{\mathbb{R}^k}) \circ \widetilde{\psi}_i, V_i) \mid i \in A\},$$

  and it follows that $E^* \to M$ is a vector bundle of rank k

Exercise

Show that $E \to M$ and $(E^*)^* \to M$ are **isomorphic** as vector bundles.

If we know the transition functions for the local trivializations of $E \to M$, we also know the transition functions of the dual local trivializations of $E^* \to M$:

Lemma

The **transition functions** of $E^* \to M$ are given by

$$\widetilde{\psi}_i \circ \widetilde{\psi}_j^{-1} : (p, w) \mapsto \left(p, (A_p^{-1})^T w\right)$$

for all $p \in \pi_E(V_i)$, where $A : \pi_E(V_i) \to \mathrm{GL}(n)$ is given by the transition functions of $E \to M$,

$$\psi_i \circ \psi_j^{-1} : (p, v) \mapsto (p, A_p v).$$

Proof: Exercise!

The most important example of a dual vector bundle for this course is the dual to the **tangent bundle** of a smooth manifold:

Definition

The vector bundle

$$T^*M := (TM)^* \to M$$

is called the **cotangent bundle** of M. Pointwise we denote $T_p^*M = (TM)_p^*$ for all $p \in M$. As for the tangent bundle we identify for any $U \subset M$ open and $p \in U$ the vector spaces $T_p^*U \cong T_p^*M$ via the inclusion map.

In order to gain a better understanding of the cotangent bundle $T^*M \to M$, we must take a deeper look at its local trivializations & charts of the total space T^*M **induced** by local coordinates on the base manifold M: (see next page)

- let $\varphi = (x^1, \dots, x^n)$ be a local coordinate system on $U \subset M$
- $\rightsquigarrow$ want to use φ to define a **local coordinate system** on $\pi_{T^*M}^{-1}(U) \subset T^*M$ compatible with the bundle structure
- **compatible** with bundle structure := charts of the total space composed with $\varphi^{-1} \times \mathrm{id}_{\mathbb{R}^n}$ from the right must be define **local trivializations**
- define **candidates** for local coordinate systems

$$\widetilde{\psi} : \pi_{T^*M}^{-1}(U) \to \varphi(U) \times \mathbb{R}^n,$$

$$\widetilde{\psi} : \omega_p \mapsto \left(\varphi(\pi_{T^*M}(\omega_p)), \omega_p \left(\left. \frac{\partial}{\partial x^1} \right|_p \right), \dots, \omega_p \left(\left. \frac{\partial}{\partial x^n} \right|_p \right) \right)$$

- verify: with $\psi := (\varphi \circ \pi, d\varphi)$ induced local coordinate system on $\pi_{TM}^{-1}(U) \subset TM$ [note: interpret codomain of $d\varphi_p$ as $\mathbb{R}^n$ via canonical identification], obtain

$$\omega_p(v_p) = \langle \mathrm{pr}_{\mathbb{R}^n}(\widetilde{\psi}(\omega_p)), \mathrm{pr}_{\mathbb{R}^n}(\psi(v_p)) \rangle \quad \forall \omega_p \in T_p^*M, v_p \in T_pM,$$

$\langle \cdot, \cdot \rangle$ = **Euclidean scalar product** in canonical coordinates

- hence: transition functions of the $\widetilde{\psi}$ are of the form

$$\widetilde{\psi}_i \circ \widetilde{\psi}_j^{-1} : (p, w) \mapsto \left(p, (d(\varphi_i \circ \varphi_j^{-1})_p^{-1})^T w\right)$$
$$= \left(p, (d(\varphi_j \circ \varphi_i^{-1})_{\varphi_i \circ \varphi_j^{-1}(p)})^T w\right),$$

  and thus **smooth**
- furthermore, $(\varphi^{-1} \times \mathrm{id}_{\mathbb{R}^n}) \circ \widetilde{\psi}$ define local trivializations (on level of sets) with smooth **matrix part**
- $\rightsquigarrow$ $\widetilde{\psi}$ define **smooth structure** on total space T^*M by vector bundle chart lemma ✓

Remark: The right hand side of $\omega_p(v_p) = \langle \mathrm{pr}_{\mathbb{R}^n}(\widetilde{\psi}(\omega_p)), \mathrm{pr}_{\mathbb{R}^n}(\psi(v_p))\rangle$ is **independent** of the chosen local coordinate system φ on M covering $p \in M$.

Understanding **sections** in the cotangent bundle is, as for vector fields, of critical importance when studying differential geometry.

Definition

Sections in $T^*M \to M$ are called **1-forms** and denoted by

$$\Omega^1(M) := \Gamma(T^*M).$$

For $U \subset M$ open, sections in $\Gamma(T^*M|_U)$ are denoted by $\Omega^1(U)$ and called **local 1-forms**.

We can easily obtain examples of 1-forms by taking the differential of smooth functions:

Example

Let $f \in C^\infty(M)$. Then the **differential of** f, $df \in \Omega^1(M)$, is given by

$$df : p \mapsto df_p.$$

(continued on next page)

Example (continuation)

In local coordinates $(x^1, \dots, x^n)$ we have $df\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial f}{\partial x^i}$ for all $1 \leq i \leq n$. This implies that df can locally be written as

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i.$$

In particular it follows for $f = x^j$ that the **coordinate 1-forms** dx^j fulfil $dx^j\left(\frac{\partial}{\partial x^i}\right) \equiv \delta^j_i$ on the domain of definition of the local coordinates. **[** this is the "global" (on chart neighbourhoods) version of $dx^j_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = \delta^j_i$ **]**

Recall that we have shown that for local coordinates $(x^1, \dots, x^n)$ of M covering $p \in M$, the set of vectors $\left\{ \left.\frac{\partial}{\partial x^i}\right|_p \,\middle|\, 1 \leq i \leq n \right\}$ is a basis of T_pM. A similar result holds for each covector space T^*_pM.

Lemma

Let $\varphi = (x^1, \ldots, x^n)$ be local coordinates defined on $U \subset M$ and let $p \in U$ be arbitrary but fixed. Then

$$\{dx_p^i \mid 1 \leq i \leq n\}$$

is a **basis** of $T_p^* M$. It is precisely the **dual basis** to the basis $\left\{ \frac{\partial}{\partial x^i}\Big|_p \;\middle|\; 1 \leq i \leq n \right\}$ of $T_p M$. Any local 1-form $\omega \in \Omega^1(U)$ can be written as

$$\omega = \sum_{i=1}^n f_i dx^i$$

with **uniquely determined** smooth functions $f_i \in C^\infty(U)$ for $1 \leq i \leq n$.

Proof:

- $\{dx_p^i \mid 1 \leq i \leq n\}$ being a basis of $T_p^* M$ that is dual to $\left\{ \frac{\partial}{\partial x^i}\Big|_p \;\middle|\; 1 \leq i \leq n \right\}$ follows from $dx_p^i \left(\frac{\partial}{\partial x^j}\Big|_p \right) = \delta_j^i$
(continued on next page)

(continuation of proof)

- observe: for any $f_i \in C^\infty(U)$, $1 \leq i \leq n$, the right hand side of $\omega = \sum_{i=1}^{n} f_i dx^i$ is a **local section** of T^*M
- this follows from the construction of the smooth manifold structure on the total space T^*M via charts of the form $\widetilde{\psi} = (\varphi \circ \pi, d\varphi^*)$ where φ is a chart on M, $d\varphi$ denotes the **vector part** of $d\varphi$, and * the pointwise dual linear map, which in particular implies that **each** dx^i **is a local 1-form**
- on the other hand: for a **given local 1-form** ω define

$$\omega_i := \omega\left(\frac{\partial}{\partial x^i}\right) \quad \forall 1 \leq i \leq n$$

- it now **suffices** to show that $\omega_i \in C^\infty(U)$ and, after that, to define $f_i := \omega_i$
- ω_i being a local smooth function follows from observing that $\omega_i \circ \varphi^{-1}$ is precisely the i**-th entry in the vector part** of $\widetilde{\psi} \circ \omega \circ \varphi^{-1}$ and thereby by definition a smooth map
- **uniqueness** of the f_i can be shown as follows: (next page)

(continuation of proof)

- suppose that locally

$$\omega = \sum_{i=1}^{n} f_i dx^i = \sum_{i=1}^{n} \widetilde{f}_i dx^i$$

such that for **at least one** $1 \leq j \leq n$, $f_j \neq \widetilde{f}_j$

- choose $p \in U$, such that $f_j(p) \neq \widetilde{f}_j(p)$ and calculate

$$\left(\sum_{i=1}^{n} f_i dx^i \right) \left(\frac{\partial}{\partial x^j}\bigg|_p \right) = f_j(p) \neq \widetilde{f}_j(p) = \left(\sum_{i=1}^{n} \widetilde{f}_i dx^i \right) \left(\frac{\partial}{\partial x^j}\bigg|_p \right)$$

which is a contradiction □

We have constructed the dual bundle $T^*M \to M$ to $TM \to M$, and we have studied their respective local sections.
Question: Can we interpret their sets of sections, that is **1-forms and vector fields**, as "dual" to each other in a meaningful way?
Answer: Yes! (see next page)

Proposition

$\Omega^1(M)$ is **isomorphic as a $C^\infty(M)$-module** to the $C^\infty(M)$**-module dual** to $\mathfrak{X}(M)$, i.e.

$$\Omega^1(M) \cong \mathrm{Hom}_{C^\infty(M)}(\mathfrak{X}(M), C^\infty(M)).$$

Proof (sketch):

- similar to the proof of $\mathrm{Der}(C^\infty(M)) \cong \mathfrak{X}(M)$ explicitly start with candidates $C^\infty(M)$-module isomorphism, given by

$$\alpha \mapsto (A_\alpha : \mathfrak{X}(M) \to C^\infty(M), \\ X \mapsto \alpha(X),\ A(X)(p) := \alpha_p(X_p) \quad \forall p \in M)$$

  for any given $\alpha \in \Omega^1(M)$
- then need to show above is well defined, use **bump functions subordinate to local charts**
- also need to describe its **inverse**
- for **details** see lecture notes! □

Remark

In practice, 1-forms are **much easier to deal with** than vector fields.This is due to how they **transform** under a change of local coordinates, which in "typical" calculations has the effect that the Jacobi matrix of the coordinate transformation does not need to be **inverted**.

Let us for example consider polar coordinates on $\mathbb{R}^2$: (next page)

Example

The change of coordinates **from polar coordinates** (r, φ) **to Cartesian coordinates** (x, y) on $\mathbb{R}^n \setminus \{y = 0,\ x \leq 0\}$ is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos(\varphi) \\ r\sin(\varphi) \end{pmatrix}.$$

Any 1-form ω on $\mathbb{R}^2$ can be (globally) written as $\omega = f(x, y)dx + g(x, y)dy$, where f, g are smooth functions **[careful:** in a more general setting we would be precise and state that f and g are **coordinate representations** of smooth functions, so that $f \circ (x, y)$ resp. $g \circ (x, y)$ are smooth functions on the manifold $\mathbb{R}^2$**]**. Then ω is in polar coordinates of the form

$$\begin{aligned}
\omega &= f(r\cos(\varphi), r\sin(\varphi))d(r\cos(\varphi)) \\
&\quad + g(r\cos(\varphi), r\sin(\varphi))d(r\sin(\varphi)) \\
&= f(r\cos(\varphi), r\sin(\varphi))(\cos(\varphi)dr - r\sin(\varphi)d\varphi) \\
&\quad + g(r\cos(\varphi), r\sin(\varphi))(\sin(\varphi)dr + r\cos(\varphi)d\varphi) \\
&= \widetilde{f}(r, \varphi)dr + \widetilde{g}(r, \varphi)d\varphi.
\end{aligned}$$

As with vector fields, we can also push forward and pull back 1-forms:

Definition

Let $F : M \to N$ be a diffeomorphism and let $\alpha \in \Omega^1(M)$, $\beta \in \Omega^1(N)$. The **pushforward** of α under F is the 1-form $F_*\alpha \in \Omega^1(N)$ given by

$$(F_*\alpha)_q := \alpha_{F^{-1}(q)} \circ d(F^{-1})_q \quad \forall q \in N.$$

The **pullback** of β under F is the 1-form $F^*\beta \in \Omega^1(M)$ given by

$$(F^*\beta)_p := \beta_{F(p)} \circ dF_p \quad \forall p \in M.$$

The above compositions denote compositions of linear maps which are given locally as matrix multiplications.

Note: An important **difference** to the pullback of vector fields is that the pullback of 1-forms in $\Omega^1(N)$ is well defined even if $F : M \to N$ is **not** a diffeomorphism, but an **arbitrary smooth map**.

Next we will generalize the vector space constructions of the **direct sum** and the **tensor product** to vector bundles.

Definition

Let $\pi_E : E \to M$ be a vector bundle of rank k and $\pi_F : F \to M$ a vector bundle of rank ℓ over an n-dimensional smooth manifold M. The **Whitney**[a] **sum of E and F** is the the direct sum of the two vector bundles $\pi_{E\oplus F} : E \oplus F \to M$ with fibres

$$(E \oplus F)_p = \pi_{E\oplus F}^{-1}(p) := E_p \oplus F_p.$$

[a] Hassler Whitney (1907 – 1989)

Remark: The **structure of a vector bundle** on

$$E \oplus F = \bigsqcup_{p\in M} (E_p \oplus F_p)$$

is then explained by the **vector bundle chart lemma** and the requirement that the following maps are **local trivializations** of $E \oplus F$: (continued on next page)

(continuation of remark)

- let $\{(\psi_i^E, V_i^E) \mid i \in I\}$ and $\{(\psi_i^F, V_i^F) \mid i \in I\}$ be **coverings of local trivializations** of E and F, respectively, such that $U_i := \pi_E(V_i^E) = \pi_F(V_i^F)$ for all $i \in I$ and such that there exists an atlas $\mathcal{A} = \{(\varphi_i, U_i) \mid i \in I\}$ of M
- we require now require that with

$$\phi_i^{-1} := (\psi_i^E \oplus \psi_i^F)^{-1} \circ (\Delta_M \times \mathrm{id}_{\mathbb{R}^{k+\ell}}) :$$

$$U_i \times \mathbb{R}^{k+\ell} \cong U_i \times (\mathbb{R}^k \times \mathbb{R}^\ell) \to \bigsqcup_{p \in U_i} (E_p \oplus F_p),$$

$$(p, v, w) \mapsto (\psi_i^E)^{-1}(p, v) \oplus (\psi_i^F)^{-1}(p, w)$$

$$\forall p \in U_i,\ v \in \mathbb{R}^k,\ w \in \mathbb{R}^\ell,$$

where $\Delta_M : p \mapsto (p, p) \in M \times M$ denotes the **diagonal embedding** and

$$\mathbb{R}^k \times \mathbb{R}^\ell \ni (v, w) \mapsto \begin{pmatrix} v \\ w \end{pmatrix} \in \mathbb{R}^{k+\ell}$$

the linear isomorphism, all ϕ_i, $i \in I$, are the inverses of **local trivializations covering** $E \oplus F$

(continuation of remark)

- in order to use the **vector bundle chart lemma** we need to check that the transition functions have the required form, i.e. are smooth in the vector part with image in $\mathrm{GL}(k+\ell)$
- obtain that for all $i, j \in I$, such that $U_i \cap U_j \neq \emptyset$,

$$\phi_i \circ \phi_j^{-1}(p, v, w) = (p, \tau_{ij}^E(p)v, \tau_{ij}^F(p)w),$$

  where τ_{ij}^E and τ_{ij}^F are the **transition functions of the local trivializations of E and F**, respectively
- lastly, we simply need to define

$$\tau_{ij}^{E\oplus F}(p) := \left(\begin{array}{c|c} \tau_{ij}^E(p) & 0 \\ \hline 0 & \tau_{ij}^F(p) \end{array}\right) \in \mathrm{GL}(k+\ell)$$

  so that we can write $\phi_i \circ \phi_j^{-1}\left(p, \binom{v}{w}\right) = \left(p, \tau_{ij}^{E\oplus F}(p)\binom{v}{w}\right)$
- $\rightsquigarrow$ **all requirements** of the vector bundle chart lemma **are fulfilled** and we conclude that $E \oplus F \to M$ is, indeed, a vector bundle of rank $k+\ell$

Example

The **tangent bundle of products** of smooth manifolds fulfils $T(M \times N) \cong TM \oplus TN$.

Note: This is **not** the only example of the Whitney sum we will encounter. **[** Hint: Recall that $\mathrm{Hom}_{\mathbb{R}}(V \times V, \mathbb{R}) \cong \mathrm{Sym}^2(V^*) \oplus \Lambda^2(V^*)$ for finite dimensional real vector spaces V. **]**

Next lecture we will, similar to the direct sum, generalize the concept of the **tensor product of vector spaces** to vector bundles.

END OF LECTURE 10

Next lecture:

- tensor products of bundles
- tensor fields