Towards a better understanding of the moduli space of projective special real manifolds

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1. **Introduction & motivation**

2. **Known results**

3. **Topology of the moduli space of projective special real manifolds**

Main reference:
"*Limit geometry of complete projective special real manifolds*" (DL, 2020), arxiv:2009.12956

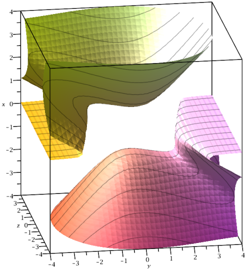
Hyperbolic homogeneous polynomials

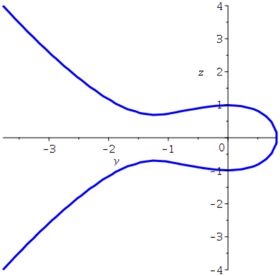
Definition

A homogeneous polynomial $h : \mathbb{R}^{n+1} \to \mathbb{R}$ is called **hyperbolic** if $\exists\, p \in \{h > 0\}$, such that $-\partial^2 h_p$ has **Minkowski signature**. Such a point p is called **hyperbolic** point of h.

- two homogeneous hyperbolic polynomials $h, \widetilde{h}$ **equivalent** $:\Leftrightarrow$ $\exists\, A \in \mathrm{GL}(n+1)$, such that $A^* \widetilde{h} = h$
- there is precisely **one** equivalence class of **quadratic** homogeneous hyperbolic polynomials in each dimension
- there is **no general classification** for higher degree $\deg(h) \geq 3$

Example: $h = x^4 - x^2(y^2 + z^2) - \frac{2\sqrt{2}}{3\sqrt{3}} xy^3$



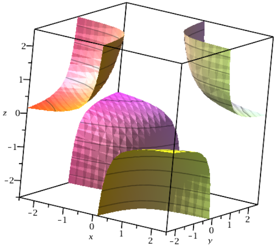


Definition

A **projective special real** (**PSR**) manifold is a hypersurface $\mathcal{H}$ contained in the level set $\{h = 1\}$ of a **cubic** homogeneous hyperbolic polynomial, such that $\mathcal{H}$ consists only of hyperbolic points of h.

- two PSR manifolds $\mathcal{H}, \widetilde{\mathcal{H}}$ **equivalent** $:\Leftrightarrow \exists\, A \in \mathrm{GL}(n+1)$, such that $A(\mathcal{H}) = \widetilde{\mathcal{H}}$
- $\mathcal{H} \subset \{h = 1\}, \widetilde{\mathcal{H}} \subset \{\widetilde{h} = 1\}$ equivalent $\Rightarrow h, \widetilde{h}$ equivalent, the **converse** is in general **not true**
- PSR manifolds have Riemannian **centro-affine fundamental form** $g = -\partial^2 h|_{T\mathcal{H}\times T\mathcal{H}}$, defined by **c.-a. Gauß eqn.** $D_X Y = \nabla^{\mathrm{ca}}_X Y + g(X,Y)\xi$ $\forall X, Y \in \mathfrak{X}(\mathcal{H})$, where ξ is the **position vector field**

Example: $h = xyz$



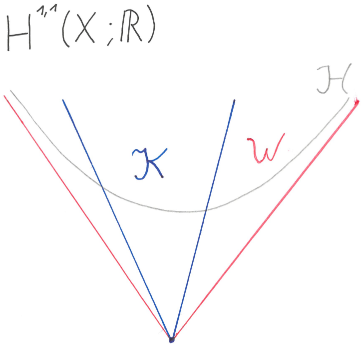
Why study PSR manifolds?

Geometry of Kähler cones [DP'04, W'04, TW'11]**:**

- for X a compact Kähler 3-fold, the cubic homogeneous polynomial

$$h : H^{1,1}(X;\mathbb{R}) \to \mathbb{R}, \quad [\omega] \mapsto \int_X \omega^3,$$

  is **hyperbolic** since every point in the **Kähler cone** $\mathcal{K} \subset H^{1,1}(X;\mathbb{R})$ is hyperbolic by the **Hodge-Riemann bilinear relations**
- $\mathcal{H} := \{h = 1\} \cap \mathcal{K}$ is a **PSR manifold**
- in general, $\mathcal{H}$ is not a **connected component** of $\{h = 1\} \cap \{\text{hyp. points of } h\}$



Why study PSR manifolds?

Explicit constructions of special Kähler and quaternionic Kähler manifolds:

- **supergravity r-map** constructs from given **PSR manifold** $\mathcal{H}$ a **projective special Kähler (PSK) manifold** $M \cong \mathbb{R}^{n+1} + i\,\mathbb{R}_{>0} \cdot \mathcal{H}$ [DV'92, CHM'12]
- **supergravity c-map** constructs from given **PSK manifold** M a (non-compact) **quaternionic Kähler manifold** $N \cong M \times \mathbb{R}^{2n+5} \times \mathbb{R}_{>0}$ [FS'90]
- above constructions **preserve geodesic completeness**

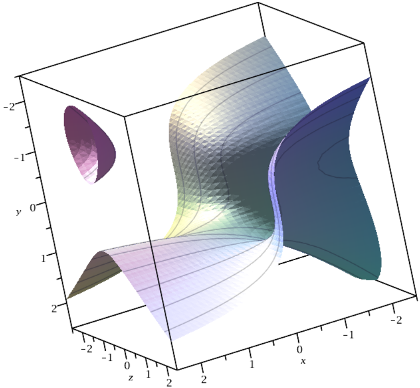
$$\begin{tikzcd}
 & \widetilde{M} \arrow[r, "\text{rigid c-map}"'] \arrow[d, "\text{ASK/PSK}"] & \widetilde{N} \arrow[d, "\text{HK/QK}"] \\
\mathcal{H} \arrow[ur, dashed] \arrow[r, "\text{sugra r-map}"'] & M \arrow[r, "\text{sugra c-map}"'] & N
\end{tikzcd}$$

Why is it difficult to classify PSR manifolds?

- set of hyperbolic polynomials is **open** in $\mathrm{Sym}^3(\mathbb{R}^{n+1})^*$
- $\dim(\mathrm{Sym}^3(\mathbb{R}^{n+1})^*)$ growth **cubically** in n while $\dim(\mathrm{GL}(n+1))$ growth only **quadratically** in n
- $\mathrm{GL}(n+1)$, acting via linear change of coordinates, is **non-compact**
- in general **polynomial equivalence** $\not\Rightarrow$ **PSR equivalence**:

Example

$\{h = x(y^2 - z^2) + y^3 = 1\}$ has **four** hyperbolic connected components, **two** of which are equivalent [CDL'14, Thm. 2,5)].



Known classification results

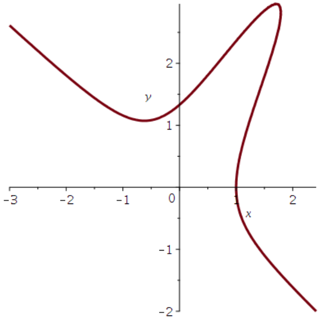
By **restricting** considered polynomials, obtain following classifications:

- **homogeneous** PSR manifolds in all dimensions [DV'92]
- **PSR curves** & **surfaces** [CHM'12, CDL'14]
- PSR manifolds with **reducible** defining polynomial [CDJL'17]

Question: What is a **realistic** approach to better understand the **moduli space** $\mathrm{Sym}^3_{\mathrm{hyp}}(\mathbb{R}^{n+1})^*/\mathrm{GL}(n+1)$ for arbitrary n?

Idea:

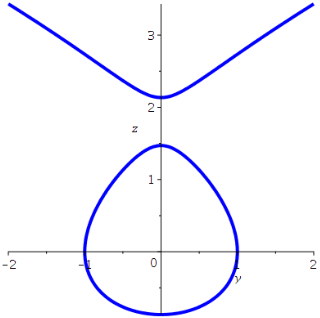
- instead of $\mathrm{Sym}^3_{\mathrm{hyp}}(\mathbb{R}^{n+1})^*/\mathrm{GL}(n+1)$, consider classes of **maximal connected PSR manifolds**, i.e. connected components of $\{h = 1\} \cap \{\text{hyp. points of } h\}$
- further split up their study in **closed** and **not closed** (in the ambient space) PSR manifolds

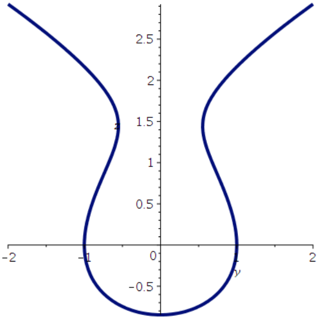


Theorem [CNS'16]

A PSR manifold is **closed** in its ambient space iff it is **complete** wrt. its centro-affine fundamental form.

[Wu'74, L'19] $\rightsquigarrow$ $\mathcal{H}$ **closed** $\Leftrightarrow$ intersection of cone $\mathbb{R}_{>0} \cdot \mathcal{H}$ with any $p + T_p\mathcal{H}$ **precompact**





Technical results

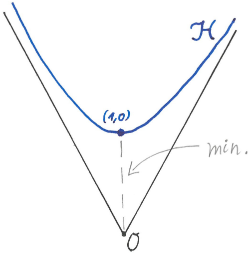
We define a convenient **standard form** for PSR manifolds. Denote $y = (y_1, \ldots, y_n)$.

Proposition [L'19]

For $\mathcal{H} \subset \{h = 1\}$ a PSR manifold & $p \in \mathcal{H}$ **arbitrary**, $\exists\, A(p) \in \mathrm{GL}(n+1)$, s.t.

- $A(p) \cdot (1, 0, \ldots, 0)^T = p$,
- $A(p)^* h = x^3 - x\langle y, y\rangle + P_3(y)$.

- $A : \mathcal{H} \to \mathrm{GL}(n+1)$ can be chosen to be **smooth**
- **explicit description** of A known, not "too bad"
- $P_3 : \mathbb{R}^n \to \mathbb{R}$ is some **cubic homogeneous polynomial**
- P_3 is **never** uniquely determined by $\mathcal{H}$
- if $\mathcal{H}$ is connected, in standard form, & $(1,0) \in \mathcal{H}$, the point $(x, y) = (1, 0)$ **minimizes the Euclidean distance** of $\mathcal{H}$ and $0 \in \mathbb{R}^{n+1}$



A generating set for moduli space of closed connected PSRs

Let $\|\cdot\|$ denote the **norm** $\|P\| := \max\limits_{\langle y,y\rangle=1} |P(y)|$ on $\mathrm{Sym}^3(\mathbb{R}^n)^*$.

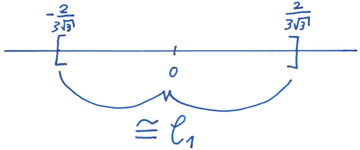
Theorem [L'19]

The connected component of $\mathcal{H} \subset \{x^3 - x\langle y,y\rangle + P_3(y) = 1\}$ containing $(x,y) = (1,0)$ is a **closed PSR manifold** iff $\|P_3\| \leq \frac{2}{3\sqrt{3}}$.

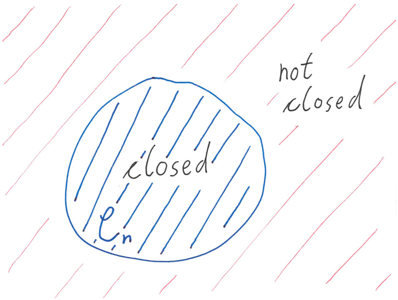
Proof: $\rightsquigarrow$ reduce problem to $\|P_3\| = \frac{2}{3\sqrt{3}}$ + "starshape" property $\rightsquigarrow$ further reduce to **dimension** 2 $\rightsquigarrow$ can use [CDL'14] and check **by hand** $\square$

Corollary

$\mathcal{C}_n := \{x^3 - x\langle y,y\rangle + P_3(y) \mid \|P_3\| \leq \frac{2}{3\sqrt{3}}\} \subset \mathrm{Sym}^3(\mathbb{R}^{n+1})^*$ is a **compact convex generating set** of the moduli space of closed connected PSR manifolds in dimension $n \geq 1$.



Consequences for the $\mathrm{GL}(n+1)$-orbits



For a given **closed connected** PSR manifold in standard form $\mathcal{H} \subset \{h = 1\}$, let $\mathrm{GL}_{\mathcal{H}}(n+1)$ denote the transformations preserving the standard form.

Corollary

The set $\mathrm{GL}_{\mathcal{H}}(n+1) \cdot h \subset \mathcal{C}_n$ is **precompact** in $\mathrm{Sym}^3(\mathbb{R}^{n+1})^*$.

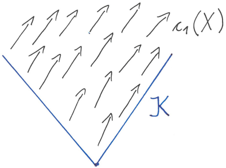
Questions: What are the possible **boundary points** $\partial(\mathrm{GL}_{\mathcal{H}}(n+1) \cdot h)$? What **information** for $\mathcal{H}$ do they give us? How can we **calculate** them?

Definition

Closed connected PSR manifolds $\overline{\mathcal{H}} \subset \{\overline{h} = 1\}$ in standard form with $\overline{h} \in \partial(\mathrm{GL}_{\mathcal{H}}(n+1) \cdot h)$ are called **limit geometries** of $\mathcal{H} \subset \{h = 1\}$.

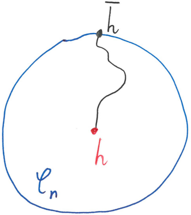
Motivation from geometry of **Kähler cones**:

- view $c_1(X)$ as **constant vector field** in $H^{1,1}(X;\mathbb{R})$
- project $c_1(X)$ **centrally** to $\mathcal{H} \subset \{h = \int_x \omega^3 = 1\}$
- calculate **standard form** of h along integral curve



In **general setting**:

- instead of $c_1(X)$, allow **any** constant vector field in ambient space $\mathbb{R}^{n+1}$
- renormalize if necessary for integral curve to **leave every compact subset** of $\mathcal{H}$
- limit geometry for choice of vector field corresponds to **limit of standard forms** $\overline{h}$ of defining polynomial h **along integral curve**



Theorem [L'20]

Limit geometries are indeed **well defined** and the **space of all possible limit geometries** grows only quadratically in n.

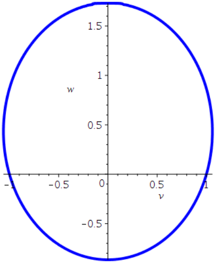
In the generic case we have the following result:

Proposition [L'20]

Let $\mathcal{H} \subset \{h = 1\}$ be a closed connected PSR manifold in standard form with $h \in \operatorname{int}(\mathcal{C}_n)$. Then **every limit geometry** of $\mathcal{H}$ is equivalent to the homogeneous space $\mathbb{R}^{n-1} \ltimes \mathbb{R}_{>0}$ corresponding to the defining polynomial

$$\overline{h} = x^3 - x(\langle v,v\rangle + w^2) + \frac{1}{\sqrt{3}}\langle v,v\rangle w + \frac{2}{3\sqrt{3}}w^3, \quad v = (v_1,\ldots,v_{n-1}).$$

$(\mathbb{R}_{>0}\cdot\overline{\mathcal{H}}) \cap ((1,0) + T_{(1,0)}\overline{\mathcal{H}})$:

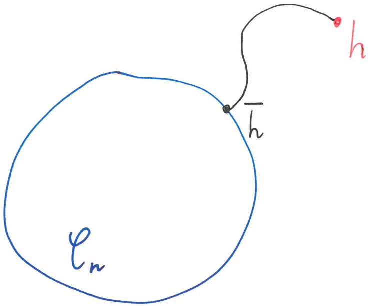


Limit geometries of non-closed maximal PSR manifolds

Question: Which properties can we expect of the boundary of orbits $\mathrm{GL}_{\mathcal{H}}(n+1)\cdot h$ for $\mathcal{H}$ **non-closed**, but still a **connected component** of $\{h=1\}\cap\{\text{hyp. points of } h\}$?

Conjecture

With $\mathcal{H}$ as above, $\partial(\mathrm{GL}_{\mathcal{H}}(n+1)\cdot h)\cap\mathcal{C}_n \neq \varnothing$.



Outlook & open questions

- apply results to geometry of manifolds in images of **r- & q=c∘r-map**
- find possible applications to the theory of the (volume-normalized) **Kähler-Ricci flow**
- "**chain**" limit geometries, obtain invariant for PSR manifolds of minimal no. of chained limit geometries to get to **homogeneous space**
  [in dim. 2, **every** limit geometry is a homogeneous space **]**
- for a better understanding of moduli space without restricting to specific connected components of $\{h = 1\}$, need **method to count hyperbolic components** of $\{h = 1\}$

Thank you for your attention!

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