Limit geometry of projective special real manifolds

David Lindemann

University of Hamburg
Department of Mathematics
Analysis and Differential Geometry

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We start with some **basic definitions**:

Definition

A **homogeneous polynomial** $h : \mathbb{R}^{n+1} \to \mathbb{R}$, $n \geq 1$, of degree at least 2 is called **hyperbolic** if $\exists p \in \{h > 0\}$, such that $-\partial^2 h_p$ is of **Lorentzian** type. Such a point p is called **hyperbolic point** of h.

- **set** of hyperbolic points of h denoted $\mathrm{hyp}(h)$
- two hyperbolic polynomials of the same degree $h, \overline{h} : \mathbb{R}^{n+1} \to \mathbb{R}$ are called **equivalent** if they are **related by a linear change of coordinates** of $\mathbb{R}^{n+1}$, i.e. if $\exists A \in \mathrm{GL}(n+1)$, such that $A^* \overline{h} = h$
- there exists precisely **one** hyperbolic homogeneous **quadratic** polynomial up to equivalence:

  $$h = x^2 - \langle y, y \rangle,$$

  where $\left(\begin{smallmatrix} x \\ y \end{smallmatrix} \right) = (x, y_1, \ldots, y_n)^T$ denote linear coordinates on $\mathbb{R}^{n+1}$ and $\langle \cdot, \cdot \rangle$ denotes the induced Euclidean scalar product on $\mathbb{R}^n$
- for **higher degrees** of h, a general classification up to equivalence in arbitrary dimensions is **not known**
- this talk: focus on $\deg(h) = 3$, that is **hyperbolic cubics**

We are interested in certain **hypersurfaces** in $\{h > 0\}$:

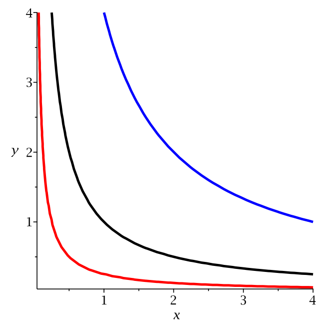
Definition

Let $h : \mathbb{R}^{n+1} \to \mathbb{R}$ be a hyperbolic cubic. A smooth hypersurface $\mathcal{H} \subset \{h = 1\}$ is called **projective special real (PSR) manifold** if $\mathcal{H} \subset \mathrm{hyp}(h)$.

- two PSR manifolds $\mathcal{H} \subset \{h = 1\}$, $\overline{\mathcal{H}} \subset \{\overline{h} = 1\}$ of dim. n are called **equivalent** if they are related by a **linear transformation of their ambient space**, i.e. if $\exists A \in \mathrm{GL}(n+1)$, such that $A(\mathcal{H}) = \overline{\mathcal{H}}$ **[**note: $\Rightarrow A^*\overline{h} = h$**]**
- by **Euler's Homogeneous Function Theorem**, $dh_p(p) = 3h(p)$ for all $p \in \mathbb{R}^{n+1}$, implying that the **position vector field** $\xi \in \mathfrak{X}(\mathbb{R}^{n+1})$ is **transversal** along a PSR manifold $\mathcal{H} \subset \{h = 1\}$
- $\rightsquigarrow$ PSR manifolds are naturally **centro-affine hypersurfaces**, with **centro-affine fundamental form** given by $g_{\mathcal{H}} = -\frac{1}{3}\partial^2 h|_{T\mathcal{H}\times T\mathcal{H}}$
- $g_{\mathcal{H}}$ is determined by the **centro-affine Gauß equation** $D_X Y = \nabla^{\mathrm{ca}}_X Y + g_{\mathcal{H}}(X,Y)\xi$ for all $X, Y \in \mathfrak{X}(\mathcal{H})$, where D is the **flat connection** on $\mathbb{R}^{n+1}$ and ∇^{ca} is the **centro-affine connection** on $\mathcal{H}$
- since $p \in \mathcal{H} \Rightarrow p \in \mathrm{hyp}(h)$, $-\partial^2 h_p$ Lorentzian and $T_p\mathcal{H} = \ker dh_p$ $\Rightarrow g_{\mathcal{H}}$ is a **Riemannian metric**

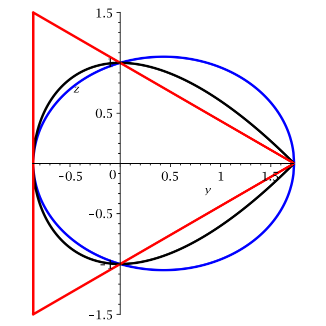
- if $\mathcal{H} \subset \mathbb{R}^{n+1}$ is a **connected component** of $\mathrm{hyp}_1(h) := \mathrm{hyp}(h) \cap \{h = 1\}$, the cone $\mathbb{R} \cdot \mathcal{H}$ is **convex**
- for any CCPSR **[c**losed & **c**onnected **PSR]** manifold $\mathcal{H}$ and all $p \in \mathcal{H}$, $(\mathbb{R}_{>0} \cdot \mathcal{H}) \cap (p + T_p\mathcal{H})$ is **precompact** in $\mathbb{R}^{n+1}$ [CNS'16]

Examples



$$\mathcal{H} = \{xyz = 1,\ x > 0,\ y > 0,\ z > 0\},$$

$$\mathcal{H} \cap \begin{cases} \{z = 1/4\} \\ \{z = 1\} \\ \{z = 4\} \end{cases}$$



$$\mathcal{H}_t = \{x^3 - x(y^2 + z^2) - \tfrac{2t}{3} yz^2 + \tfrac{2}{3\sqrt{3}} y^3 = 1,\ x \geq 1\},$$

$$\partial((\mathbb{R}_{>0} \cdot \mathcal{H}_t) \cap (p + T_p\mathcal{H})),$$

$$p = (1,0,0),\ t = \sqrt{3},\ t = 0,\ t = -\sqrt{3}/2$$

Why study PSR manifolds?

- **supergravity**: construction of the **supergravity r-map** corresponding to dimensional reduction from 5 to 4 spacetime dimensions [DV'92, CHM'12]:

$$\text{r-map}: \{\text{connected PSR mfs. of dimension } n \quad \mathcal{H}\}$$
$$\longrightarrow \{\text{connected PSK mfs. of dimension } 2n+2 \quad M\},$$

"PSK" = "projective special Kähler"

- **quaternionic Kähler geometry**: **supergravity c-map** obtained by dimensional reduction from 4 to 3 spacetime dimensions [FS'90]:

$$\text{c-map}: \{\text{connected PSK mfs. of dimension } 2n+2 \quad M\}$$
$$\longrightarrow \{\text{connected QK mfs. of dimension } 4n+8 \quad N\},$$

"QK" = "quaternionic Kähler"

$$\mathcal{H} \xrightarrow[\text{r-map}]{} M \cong \mathbb{R}^{n+1} + i\,\mathbb{R}_{>0}\cdot\mathcal{H} \xrightarrow[\text{c-map}]{} N \cong M \times \mathbb{R}^{2n+5} \times \mathbb{R}_{>0}$$

QK-manifolds constructed via the $c \circ r =: q$-map have **negative scalar curvature**

- r- & c-map preserves **completeness** [CHM'12], allows construction of explicit locally inhomogeneous complete non-compact quaternionic Kähler manifolds [CDJL'17]

PSR manifolds also appear in the study of **compact Kähler manifolds** of **cx. dim** 3:

- for a compact Kähler 3-fold X, define a **homogeneous cubic**

$$h : H^{1,1}(X,\mathbb{R}) \to \mathbb{R}, \quad h([\omega]) := \int\limits_X \omega \wedge \omega \wedge \omega$$

- **Hodge-Riemann bilinear relations** $\rightsquigarrow h : H^{1,1}(X,\mathbb{R}) \cong \mathbb{R}^{h^{1,1}} \to \mathbb{R}$ is **hyperbolic**, i.e. every point in the **Kähler cone** $\mathcal{K} \subset H^{1,1}(X,\mathbb{R})$ is a hyperbolic point of h
- $\Rightarrow \mathcal{H} := \{h = 1\} \cap \mathcal{K}$ is a **PSR manifold**
- **interpretation** of geometrical data obtained this way is **far from trivial**
- e.g. it is unclear precisely **which** PSR manifolds can be **constructed** this way
- **sectional curvature bounds** of such PSR mfds. $(\mathcal{H}, g_{\mathcal{H}})$ have been studied in [W'04, TW'11]
- **higher dimensional** compact Kähler manifolds $\leftrightsquigarrow$ **generalized** PSR manifolds

Some **known results** about PSR manifolds:

Theorem

- a PSR manifold $(\mathcal{H}, g_{\mathcal{H}})$, $\mathcal{H} \subset \{h = 1\} \subset \mathbb{R}^{n+1}$, is **geodesically complete** if and only if it is **closed in its ambient space** $\mathbb{R}^{n+1}$ [CNS'16]
- $\rightsquigarrow$ above justifies identifying **complete connected PSR manifolds** with **CCPSR manifolds**
- **homogeneous PSR manifolds** have been classified in [DV'92]
- **PSR curves** have been classified up to equivalence in [CHM'12], have **2 complete PSR curves** and **1 maximal incomplete PSR curve**
- **PSR surfaces** have been classified up to equivalence in [CDL'14], have **5 "isolated" complete connected PSR surfaces** and **1 one-parameter family of pairwise inequivalent PSR surfaces**

Remarks:

- a **general classification** of PSR manifolds up to equivalence is **unknown** at this point
- whether a similar **completeness result** holds for generalized PSR mfds. $\mathcal{H} \subset \{h = 1\}$ is an **open problem** for **all** $\deg(h) \geq 4$ in dimension ≥ 2

Question: Is there a **reasonable way** to define a notion of **"limit geometry"** for **CC**PSR manifolds?

Answer: Yes, use ideas from **Kähler-Ricci flow** and results about **structure of the moduli set** of CCPSR manifolds!

- the equation of the **Kähler-Ricci flow on classes** in the Kähler cone $\mathcal{K}$ of a compact Kähler 3-fold X is given by

$$\partial_t[\omega_t] = 2\pi c_1(X)$$

- $c_1(X) = \frac{1}{2\pi}[\mathrm{Ric}(\omega)] \in H^{1,1}(X,\mathbb{R})$ is the first **Chern class**, right hand side **independent** of $\omega \in \mathcal{K}$
- **solution** is thus an **affine line** in $\mathcal{K}$ if $c_1(X) \neq 0$
- the **volume preserving** K.-R. flow on classes is obtained by the **central projection** of $c_1(X)$ to $T\mathcal{H}$, viewed as **constant vector field** in $H^{1,1}(X,\mathbb{R})$,

$\rightsquigarrow$ this motivates the following **ansatz**:

Ansatz 1

A **limit geometry** of a CCPSR mfd. $\mathcal{H} \subset \{h = 1\} \subset \mathbb{R}^{n+1}$ should correspond to **evolution of geometrical data** along curves $\gamma : [0,1) \to \mathcal{H}$ that are, up to reparametrisation, **integral curves of central projections of constant vector fields** on $\mathbb{R}^{n+1}$ to $T\mathcal{H}$ and **leave all compact sets** as $t \to 1$.

$\rightsquigarrow$ need to **study** $g_{\mathcal{H}}|_{\gamma(t)}$ **as** $t \to 1$

Problem: How?

$\rightsquigarrow$ want to **actually calculate something**, need the following:

Proposition [L'19]

Let $\mathcal{H} \subset \{h = 1\} \subset \mathbb{R}^{n+1}$ be a **CCPSR** manifold. Then $\exists$ a smooth map

$$A : \mathcal{H} \to \mathrm{GL}(n+1),$$

such that for all $p \in \mathcal{H}$, $A(p) \cdot \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = p$, where $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = (1, 0, \ldots, 0)^T$, and

$$A(p)^* h = x^3 - x\langle y, y\rangle + P_3(y),$$

where $(x, y) = (x, y_1, \ldots, y_n)$ denote **linear coordinates** on the ambient space $\mathbb{R}^{n+1}$, $\langle \cdot, \cdot \rangle$ is the standard **Euclidean scalar product**, and $P_3 : \mathbb{R}^n \to \mathbb{R}$ is a homogeneous cubic.

Remarks:

- call $h = x^3 - x\langle y, y\rangle + P_3(y)$ **standard form**
- if h is in standard form and $\mathcal{H}$ is closed and connected, $\mathcal{H}$ is the **connected component** of $\{h = 1\}$ containing $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$
- the map A is unique only up to $\mathrm{O}(n)$, as orthogonal transformations in the y-coordinates will preserve h in standard form **[**preserve = standard form, usually a **different one**!**]**
- P_3 is **never** uniquely determined by $\mathcal{H}$
- one might study the **infinitesimal change** of the P_3-term as $p \in \mathcal{H}$ varies, yields formula up to $\mathfrak{so}(n)$

⇝ in order to study $g_{\mathcal{H}}|_{\gamma(t)}$ as $t \to 1$, study **choice of standard form of** h w.r.t. evolving reference point $\gamma(t) \in \mathcal{H}$
⇝ **effectively:** need to study $P_3 = P_3|_{\gamma(t)}$ as $t \to 1$
To proceed we need to understand **which** polynomials P_3 correspond to a **CC**PSR manifold.

Theorem [L'19]

The connected component of $\{h = x^3 - x\langle y, y\rangle + P_3(y) = 1\} \subset \mathbb{R}^{n+1}$ containing $\binom{x}{y} = \binom{1}{0}$ is a CCPSR manifold **if and only if**

$$\|P_3\| := \max_{\langle y,y\rangle=1} P_3(y) \leq \frac{2}{3\sqrt{3}}.$$

Sketch of proof:

- use **standard form Proposition**
- reduce problem to **2-dimensional case**
- show that if $\mathcal{H} = \mathcal{H}_1$ corresponding to P_3 is **CCPSR**, $\mathcal{H}_t$ corresponding to tP_3 is **CCPSR** for all $0 \leq t \leq 1$ **[**"star-shape property"**]**
- reduce problem to **2-dimensional case** with h **singular at infinity**, i.e. $\exists\ p \in \partial(\mathbb{R}_{>0} \cdot \mathcal{H}) \setminus \{0\}$, such that $dh_p = 0$
- ⇝ **2-parameter problem**, solve by hand □

Corollary

For all $n \geq 1$, the set

$$x^3 - x\langle y, y\rangle + \mathcal{C}_n \subset \mathrm{Sym}^3(\mathbb{R}^{n+1})^*$$

with $\mathcal{C}_n = \left\{ P_3 \in \mathrm{Sym}^3(\mathbb{R}^n)^* \;\middle|\; \|P_3\| \leq \frac{2}{3\sqrt{3}} \right\}$ is a **compact convex generating set** of the **moduli set of CCPSR manifolds** of dimension n.

Remarks:

- **compactness** of $\mathcal{C}_n$ $\rightsquigarrow$ expect a **well-defined limit** "up to $\mathrm{O}(n)$" of $P_3|_{\gamma(t)}$
- for the next step, that is **calculating limits**, need to find a **good parametrisation of the curve** γ in the **most general case** and need **explicit formula for** $A(\gamma(t)) \in \mathrm{GL}(n+1)$ for all $t \in [0,1)$:

$\rightsquigarrow$ for $p = \begin{pmatrix} p_x \\ p_y \end{pmatrix} \in \mathcal{H}$ and $E(p) \in \mathrm{GL}(n)$ consider

$$A(p) := \left(\begin{array}{c|c} p_x & -\left.\frac{\partial_y h}{\partial_x h}\right|_p \circ E(p) \\ \hline p_y & E(p) \end{array} \right),$$

where $E(p)$ fulfils

$$-\frac{1}{2}\partial^2 h_p \left(\begin{pmatrix} -\frac{\partial_y h(E(p)y)}{\partial_x h} \\ E(p)y \end{pmatrix}, \begin{pmatrix} -\frac{\partial_y h(E(p)y)}{\partial_x h} \\ E(p)y \end{pmatrix} \right) = \langle y, y\rangle$$

for all $y \in \mathbb{R}^n$, and $E\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \mathbb{1}$

Since we only want a limit **up to an orthogonal transformation** and **reparametrisations of** γ, we can take the following **most general** approach:

Ansatz 2

Let $\mathcal{H}$ be the **connected component of** $\{h = 1\}$,

$$h(x,v,w) := x^3 - x\left(\langle v,v\rangle + w^2\right) + C(v) + Q(v)w + b\langle \eta, v\rangle w^2 + aw^3,$$

containing $(x,v,w) = (1,0,0)$ with $v = (v_1,\ldots,v_{n-1})$, and let $\gamma : [0,R) \to \mathcal{H}$ be given by

$$\gamma : t \mapsto h(1,0,t)^{-1/3}\begin{pmatrix}1\\0\\t\end{pmatrix}$$

where R is the **smallest positive zero** of $h(1,0,t)$. **[**we consider the limit $t \to R$**]**

Remarks:

- η is a **unit vector** w.r.t. the Euclidean scalar product in $\mathbb{R}^{n-1}$, $b \in \mathbb{R}$
- for **CC**PSR require $\max\limits_{\langle v,v\rangle + w^2 = 1}\left(C(v) + Q(v)w + b\langle \eta, v\rangle w^2 + aw^3\right) \leq \frac{2}{3\sqrt{3}}$
- **precompactness** of $(\mathbb{R}_{>0}\cdot\mathcal{H}) \cap (p + T_p\mathcal{H})$ for $\mathcal{H}$ CCPSR $\Rightarrow$ $R \in [\sqrt{3}/2, \sqrt{3}]$

Next step: Calculate!

Main difficulties:

- choosing a **"good"** transformation $A(\gamma(t)) \in \mathrm{GL}(n)$ for t near R, recall that we have the freedom of $\mathrm{O}(n)$
- otherwise, you **WILL NOT SEE** what to expect in the end
- **solution:** start with **easier case** $b = 0$, hope for the best, actually **get results**, turns out these are the most general **"limit polynomials"** you can get **in any case**
- for the actual **limit calculation**, the main issue is **how many zero eigenvalues** the positive semi-definite bilinear form

$$\langle\!\langle y, y \rangle\!\rangle := -\frac{1}{2}\partial^2 h_{(1,0,R)}\left(\begin{pmatrix} -\frac{\partial_y h(y)}{\partial_x h} \\ y \end{pmatrix}, \begin{pmatrix} -\frac{\partial_y h(y)}{\partial_x h} \\ y \end{pmatrix}\right)$$

  has and **how** you get them, from **"b too big"** or **"$1/R$ eigenvalues of Q"**
- **other big difficulty:** the **maximality condition** $\|P_3\| \leq \frac{2}{3\sqrt{3}}$, is difficult to "see" and, hence, use
- **solution: none in general**, context-sensitive as in depends on number of zero eigenvalues of $\langle\!\langle y, y \rangle\!\rangle$, mostly used to deal with C-part

Main results of [L'20]: (next page)

Theorem 1 [L'20]

- **Dimension 1:** All possible **limit polynomials** are **equivalent** to $h = x^3 - xy^2 - \frac{2}{3\sqrt{3}}y^3$.
- **Dimension 2:** All possible **limit polynomials** are **equivalent** to **either**

$$h = x^3 - x(y^2 + z^2) - \frac{1}{\sqrt{3}}y^2 z - \frac{2}{3\sqrt{3}}z^3, \quad \mathcal{H} \cong H$$

where H is the **hyperbolic plane**, or

$$h = x^3 - x(y^2 + z^2) + \frac{2}{\sqrt{3}}y^2 z - \frac{2}{3\sqrt{3}}z^3, \quad \mathcal{H} \cong \mathbb{R}^2.$$

Note: Both limit spaces are **homogeneous**.

- **Dimension $\geq$3:** All possible **limit polynomials** are **equivalent** to one of

$$\begin{aligned} h = {} & x^3 - x(\langle s,s\rangle + \langle u,u\rangle + w^2) \\ & + \left(\frac{2}{\sqrt{3}}\langle s,s\rangle - \frac{1}{\sqrt{3}}\langle u,u\rangle\right) w + \sum_{i=1}^{m} s_i \langle u, F_i u\rangle - \frac{2}{3\sqrt{3}}w^3, \end{aligned}$$

$s = (s_1, \ldots, s_m)$, $u = (u_1, \ldots, u_{n-1-m})$ for $0 \leq m \leq n-1$, where each F_i, $1 \leq i \leq n-1-m$, is a **symmetric** $((n-1-m)\times(n-1-m))$**-matrix**, such that for all $c \in \mathbb{R}^m$, $\langle c,c\rangle = 1$, the **eigenvalues of** $\sum_{i=1}^{m} c_i F_i$ **are contained in** $[-1,1]$. **Every** such polynomial defines a **CCPSR manifold** with non-compact symmetry group of dimension **at least** 1.

Proposition [L'20]

Let $(\overline{\mathcal{H}}, g_{\overline{\mathcal{H}}})$ be a **limit geometry** of a CCPSR manifold $(\mathcal{H}, g_{\mathcal{H}})$ with respect to a curve $\gamma : [0, R) \to \mathcal{H}$. Then for every compactly embedded open subset $U \subset \overline{\mathcal{H}}$ and every $\varepsilon > 0$ there exists a compactly embedded open subset $U' \subset \mathcal{H}$ and a diffeomorphism $F : U \to U'$, such that

$$\| g_{\overline{\mathcal{H}}} - F^* g_{\mathcal{H}} \|_{g_{\overline{\mathcal{H}}}} < \varepsilon.$$

in $\overline{U}$. If U contains the point $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \in \overline{\mathcal{H}}$, there exists $N \in [0, R)$ such that for all $t \in [N, R)$, U' can be chosen to contain the point $\gamma(t) \in \mathcal{H}$.

Theorem 2 [L'20]

If a CCPSR manifold $\mathcal{H} \subset \{h = 1\}$ of dimension $n \geq 3$ is **not singular at infinity**, i.e. $\forall p \in \partial(\mathbb{R}_{>0} \cdot \mathcal{H}) \setminus \{0\}$ we have $dh_p \neq 0$, **every possible limit geometry** is isomorphic to $\mathbb{R}_{>0} \ltimes \mathbb{R}^{n-1}$, corresponding to the limit polynomial

$$h = x^3 - x(\langle u, u \rangle + w^2) - \frac{1}{\sqrt{3}} \langle u, u \rangle w - \frac{2}{3\sqrt{3}} w^3. \quad \textbf{[} \text{ i.e. } m = 0 \textbf{]}$$

- CCPSR manifolds that are **not singular at infinity** correspond to the **interior of** $\mathcal{C}_n$ in the subspace topology
- Theorem 2 implies that the topology of the **moduli space** of CCPSR manifolds **[**when equipped with the quotient topology**]** is **not Hausdorff**

Question: What happens if one "**chains**" the construction of limit geometries? Does this make life **easier**?
Answer: **No**. BUT: **Yes** in certain **directions**!

Definition

Let $\mathcal{H}$ be a CCPSR manifold in standard form. The **first variation** of the corresponding P_3-term is defined by

$$\delta P_3(y) := \boldsymbol{d}\left(\left(A^* h\right)\left(\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right)\right) = -\frac{2}{3}\langle y, y\rangle\langle y, dy\rangle + dP_3\left(by + \frac{1}{4}\partial^2 P_3 \cdot dy\right).$$

- $\boldsymbol{d}$ is to be understood as the de-Rham differential of the vector-valued smooth function $\left(A^* h\right)\left(\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right) : \mathcal{H} \to \mathrm{Sym}^3(\mathbb{R}^{n+1})^*$, $p \mapsto \left(A(p)^* h\right)\left(\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right)$,
- $b \in \mathrm{Lin}(\mathbb{R}^n, \mathfrak{so}(n))$ is of the form

$$b = \sum_{i=1}^{n} a_i \otimes dy_i, \quad a_i \in \mathfrak{so}(n),$$

so that $by = \sum_{i=1}^{n} (a_i y) \otimes dy_i$.

- the b-term should be interpreted as an **infinitesimal rotation** of $\mathcal{H}$

While not necessarily being a generator of a **symmetry group**, infinitesimally changing the standard form of limit polynomials in s_ℓ-direction **preserves** said form up to the F_i-terms!

Lemma

Suppose $\dim(\mathcal{H}) \geq 3$ and $\mathcal{H}$ is a **limit geometry CCPSR manifold** with

$$P_3(y) = \left(\frac{2}{\sqrt{3}} \langle s, s \rangle - \frac{1}{\sqrt{3}} \langle u, u \rangle \right) w + \sum_{i=1}^{m} s_i \langle u, F_i u \rangle - \frac{2}{3\sqrt{3}} w^3$$

and $1 \leq m \leq n-2$. Then for $b = 0 \in \mathrm{Lin}(\mathbb{R}^n, \mathfrak{so}(n))$,

$$\delta P_3(y)(\partial_{s_\ell}) = -\langle u, u \rangle s_\ell + \sum_{i=1}^{m} s_i \langle F_i u, F_\ell u \rangle \quad \forall 1 \leq \ell \leq m.$$

- this means: infinitesimally changing the reference point in s_ℓ-direction without rotating **preserves** the limit polynomial form up to changes in the F_i-terms
- $\rightsquigarrow$ to calculate the limit geometry of $\mathcal{H}$ in s_ℓ-direction, one can, instead of **painfully** trying to work out the transformation matrix A explicitly, consider the following **ODE** (next page)

Proposition

A choice of standard form of $\mathcal{H}$ along the curve corresponding to moving in s_ℓ-direction with **constant speed** $g_{\mathcal{H}}|_{\binom{1}{0}}(\partial_{s_\ell}, \partial_{s_\ell}) = \frac{2}{3}$ is given by the corresponding $P_3 = P_3(t)$-term

$$P_3(t)(y) = \left(\frac{2}{\sqrt{3}} \langle s, s \rangle - \frac{1}{\sqrt{3}} \langle u, u \rangle \right) w + \sum_{i=1}^{m} s_i \langle u, F_i(t) u \rangle - \frac{2}{3\sqrt{3}} w^3$$

where the $F_i(t)$, $1 \leq i \leq m$, fulfil the system of ODEs

$$\begin{aligned} & \partial_t \left(P_3(t)(y) \right) = \delta P_3(t)(y)(\partial_{s_\ell}) \\ \Leftrightarrow \quad & \partial_t F_i(t) = \begin{cases} \frac{1}{2}(F_i(t) F_\ell(t) + F_\ell(t) F_i(t)), & i \neq \ell, \\ -\mathbb{1} + F_\ell(t)^2, & i = \ell. \end{cases} \end{aligned}$$

with initial condition $F_i(0) = F_i$.

- one can write down the **general solution** of the above ODE **explicitly**
- the **maximal domain** of the smooth symmetric matrix-valued functions is $\mathbb{R}$ by the **completeness** of $\mathcal{H}$, i.e. $F_i : \mathbb{R} \to \mathrm{Mat}((n-1-m) \times (n-1-m))$ for all $1 \leq i \leq m$.
- $\rightsquigarrow$ can calculate the "**second**" **limit geometry** by calculating the limits $\lim\limits_{t \to \infty} \langle u, F_i(t) u \rangle$!

We find the following **application**:

Proposition

If there exists $c = (c_1, \ldots, c_m)^T \in \mathbb{R}^m$, $\langle c, c \rangle = 1$, such that $\sum_{i=1}^{m} c_i F_i$ has **eigenvalues** contained in $[-1, 1)$ **[**that is **not** 1**]** then there exists a limit geometry of $\mathcal{H}$ corresponding to

$$P_3(y) = \left(\frac{2}{\sqrt{3}} \langle s, s \rangle - \frac{1}{\sqrt{3}} \langle u, u \rangle \right) w - \sum_{i=1}^{m} s_i \langle u, u \rangle - \frac{2}{3\sqrt{3}} w^3.$$

[the above P_3 corresponds to $F_i = -\mathbb{1}$ for all $1 \leq i \leq m$**]**

For $n = 3$, c as above can **always** be found.

Question for $n > 3$

Can we **always** find such a c?

- **Answer: No!** $\rightsquigarrow$ would mean that every possible **homogeneous** CCPSR manifold has a standard form as above, **not true**
- **However:** should be possible for $\mathcal{H}$ **inhomogeneous**!

Applications & outlook:

- limit geometry/polynomials for **maximal incomplete** PSRs!
- **Supergravity:** Calculate the **Kretschmann scalar** of **q-map image** of limit polynomials, obtain general description of **"limit geometry"** of complete QK manifolds in image of q-map
- **gluing** CCPSR manifolds with regular boundary behaviour "**at infinity**" to obtain **compact analogues**
- **Kähler-Ricci flow:** if the K-R flow is **time-incomplete** and the limit class is contained in $\partial\mathcal{P} \setminus \{0\}$, where $\mathcal{P}$ is the **positive cone** [W'04], get information of (hypothetical) limit as in **limit of the underlying Kähler** manifold, possibly via some sort of **degeneration**
- **CCPSR manifolds:** optimal **sectional & scalar curvature bounds** in dimension ≥ 3 **[note:** in dimension 2, scal is optimally bounded by $[-9/4, 0]$ in the class of CCPSR surfaces [L'18]**]**

THANK YOU FOR YOUR ATTENTION!

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