Advanced algebra
Homological algebra and representation theory
Wintersemester 2014/15

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Exercise Sheet 10

to be solved till exercise class 19.01.2015

1	Let $\mathcal{C}$ be an abelian category and $Ch_{\mathcal{C}}$ the associated category of chain complexes.	
1.1	Then the category $Ch_{\mathcal{C}}$ is additive.	◯ True /◯ False
1.2	Then the category $Ch_{\mathcal{C}}$ is abelian.	◯ True /◯ False
2	Let $C_\bullet$ a complex of abelian groups such that all groups C_n are *free* abelian groups.	
2.1	Then the subgroup of cycles Z_n is free for all n.	◯ True /◯ False
2.2	Then the subgroup of boundaries B_n is free for all n.	◯ True /◯ False
2.3	Then the homology $H_n(C)$ is free for all n.	◯ True /◯ False

3 For the following unions of triangles, let C_0, C_1, C_2 be the free abelian groups generated by vertices i, edges (ij) resp. triangles (ijk), where we set $(ij) = -(ji)$ and $(ijk) = (jki) = (kij) = -(kji) = -(ikj) = -(jik)$. Let a differential be given by $\partial(i) = 0, \partial(ij) = j - i$ and $\partial(ijk) = ij + jk + ki$. Then calculate the homology groups:

1. A plain triangle
2. A plain triangle subdivided into 4 triangles.
3. A tetrahedron and an octahedron.
4. A square (subdivided into two triangles) with both pairs of parallel sides identified.
5. A square (subdivided into two triangles) with one pair of parallel sides identified the opposite way.
6. A square (subdivided into two triangles) with one pair of parallel sides identified the opposite way and the other one identified the usual way.

What are the respective topological spaces? What happens for the tetrahedron, if we add an additional element in C_3 representing the inner of the tetrahedron?

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1. Find two different projective resolution of the $\mathbb{Z}$-module $\mathbb{Z}_n$ and find a homotopy equivalence.
2. Give a homotopy equivalence between the chain complex C_0, C_1, C_2 associated to a single triangle in question 3) and the rather trivial chain complex where $D_0 = \mathbb{Z}, C_{i>0} = 0$. Especially check that the homology groups are the same.

5 Let $C_\bullet$ and $D_\bullet$ be chain complexes in an abelian category $\mathcal{C}$ and let $\phi_\bullet : C_\bullet \to D_\bullet$ be a morphism of chain complexes.

(i) Show that

$$E_n := C_{n-1} \oplus D_n \qquad d(a,b) = (-da, \phi(a) + db) \quad \text{mit } a \in C_{n-1} \text{ und } b \in D_n$$

defines a chain complex, the so-called mapping cone $E(\phi_\bullet)$ of $\phi_\bullet$.

(ii) Show that the inclusion $\iota : D_\bullet \to E_\bullet$ is a chain map.

(iii) Show that any commutative diagram of chain maps

$$\begin{tikzcd} C_\bullet \arrow[r, "\phi_\bullet"] \arrow[d, "\psi_C"] & D_\bullet \arrow[d, "\psi_D"] \\ C'_\bullet \arrow[r, "\phi'_\bullet"] & D'_\bullet \end{tikzcd}$$

induces a chain map of the mapping cones $E(\phi_\bullet) \to E(\phi'_\bullet)$.

(iv) Show that the chain map $\phi_\bullet : C_\bullet \to D_\bullet$ induces a long exact sequence

$$\cdots \to H_n(C) \to H_n(D) \to H_n(E(\phi)) \to H_{n-1}(C) \to \cdots$$

in homology.

6 Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of abelian categories with enough projectives. Suppose that we are given an exact sequence

$$0 \to K \to P_{n-1} \to \ldots \to P_1 \to P_0 \to M \to 0$$

in $\mathcal{C}$, where all modules P_i are projective.

(i) Show that for all $i > n$ the identity

$$L_i F(M) \cong L_{i-n} F(K)$$

holds.

Hint:
There are two possible solutions: either use a projective resolution of K to construct a projective resolution of M. Or first consider the case $n = 1$ and then use induction.

(ii) Conclude that $L_1 F = 0$ implies $L_i F = 0$ for all i.

For questions or comments regarding exercise sheets or classes please contact:
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