Holonomy groups and the heterotic string

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Holonomy groups and applications in string theory

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Based on a collaboration which includes
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Outline

Killing spinor equations

A parallel transport equation for the supercovariant connection $\mathcal{D}$

$$\delta\psi_A| = \mathcal{D}_A\epsilon = \nabla_A\epsilon + \Sigma_A(e,F)\epsilon = 0$$

and possibly algebraic equations

$$\delta\lambda| = \mathcal{A}(e,F)\epsilon = 0$$

where ∇ is the Levi-Civita connection, $\Sigma(e,F)$ a Clifford algebra element

$$\Sigma(e,F) = \sum_k \Sigma_{[k]}(e,F)\Gamma^{[k]}$$

e frame and F fluxes, ϵ spinor, Γ gamma matrices.

Can the KSE be solved without any assumptions on the metric and fluxes?

ie find those (e,F) such that the KSE admit $\epsilon \neq 0$ solutions.

Spinorial geometry

The ingredients of the spinorial method to solve the supergravity KSE [J Gillard, U Gran, GP; hep-th/0410155] are

- Gauge symmetry of KSE
  It is used to choose the Killing spinor directions or their normals. Very effective for backgrounds with small and large number of solutions
- Spinors in terms of forms
- An oscillator basis in the space of Dirac spinors
  Allows to extract the geometric information using the linearity of KSE.

All three ingredients are essential for the effectiveness of the method.

Gauge symmetry and holonomy

The gauge symmetry G of the KSE are the (local) transformations such that

$$\ell^{-1}\mathcal{D}(e,F)\ell=\mathcal{D}(e^{\ell},F^{\ell})\,,\quad \ell^{-1}\mathcal{A}(e,F)\ell=\mathcal{A}(e^{\ell},F^{\ell})$$

i.e. preserve the form of the Killing spinor equations.

SUGRA	Gauge	Holonomy
$D=11$	$Spin(10,1)$	$SL(32,\mathbb{R})$
IIB	$Spin_c(9,1)$	$SL(32,\mathbb{R})$
Heterotic	$Spin(9,1)$	$Spin(9,1)$
$\mathcal{N}=1, D=4$	$Spin_c(3,1)$	$Pin_c(3,1)$

The holonomy groups have been found in [Hull, Duff, Lu, Tsimpis, GP].

- Backgrounds related by a gauge transformation are identified
- 2 generic spinors ϵ_1,ϵ_2 in D=11 and IIB have isotropy group $\text{Stab}(\epsilon_1,\epsilon_2)$ in G, $\text{Stab}(\epsilon_1,\epsilon_2)=\{1\}$

Killing spinor equations

The geometric data of $\mathcal{N} = 1$ supergravity are

- A 4-d Lorentzian manifold M, the spacetime.
- A (Hodge) Kähler manifold N with Kähler potential K which admits a holomorphic, metric preserving, group action and the associated Killing holomorphic vectors fields and moment maps are ξ and μ, respectively.
- The scalar fields ϕ are maps from M to the Kähler manifold.
- A gauge connection B over the spacetime M which gauges the holomorphic isometries of N.

The gravitino Killing spinor equation of $\mathcal{N} = 1$ supergravity is

$$2\nabla_A \epsilon_L + \frac{1}{2}(\partial_i K \mathcal{D}_A \phi^i - \partial_{\bar{i}} K \mathcal{D}_A \phi^{\bar{i}})\epsilon_L + i e^{\frac{K}{2}} W \Gamma_A \epsilon_R = 0$$

The gaugino is

$$F^a_{AB} \Gamma^{AB} \epsilon_L - 2i\mu^a \epsilon_L = 0$$

and the matter multiplet KSE is

$$i\Gamma^A \epsilon_R \mathcal{D}_A \phi^i - e^{\frac{K}{2}} G^{i\bar{j}} D_{\bar{j}} \bar{W} \epsilon_L = 0$$

where K Kähler potential, W holomorphic, μ moment map and

$$\mathcal{D}_A \phi^i = \partial_A \phi^i - B^a_A \xi^i_a$$

$Spin(3,1)$ Spinors

$Spin(3,1) = SL(2,\mathbb{C})$. The chiral and anti-chiral representations are **2** and $\bar{\mathbf{2}}$. Dirac representation $\Lambda^*(\mathbb{C}^2)$. Weyl representations $\Lambda^{\mathrm{ev}}(\mathbb{C}^2)$ and $\Lambda^{\mathrm{odd}}(\mathbb{C}^2)$.
Gamma matrices

$$\Gamma_0 = -e_2 \wedge + e_2 \lrcorner \ , \quad \Gamma_2 = e_2 \wedge + e_2 \lrcorner$$
$$\Gamma_1 = e_1 \wedge + e_1 \lrcorner \ , \quad \Gamma_3 = i(e_1 \wedge - e_1 \lrcorner)$$

The Majorana spinors are found using the reality condition $R = -\Gamma_{012} *$. The real components of the Weyl spinors 1 and $i1$ are

$$1 + e_1 \ , \quad i(1 - e_1)$$

Thus

$$\epsilon = 1 + e_1 \ , \quad \epsilon_L = 1 \ , \quad \epsilon_R = e_1$$

$N = 1$ backgrounds

Based on [U Gran, J Gutowski, GP; arXiv:0802.1779].

$Spin(3,1) = SL(2,\mathbb{C})$ has a single orbit in $\mathbb{C}^2$. So the first Killing spinor can be chosen as

$$\epsilon = 1 + e_1$$

Solving the Killing spinor equations, the spacetime admits a null, Killing, integrable vector field X

$$\nabla_{(A}X_{B)} = 0\,, \quad X \wedge dX = 0\,, \quad g(X,X) = 0$$

The spacetime metric can be written as

$$ds^2 = f du(dv + V du + w_i dx^i) + g_{rs} dx^r dx^s\,, \quad r,s = 1,2$$

where $X = \partial_v$ and $f = f(u, x^r)$. The conditions on the rest of the fields are known.

$N = 2$ backgrounds

The isotropy group of the first Killing spinor $\epsilon = \epsilon_1$ in $Spin(3,1)$ is $\mathbb{C}$. Using this, the second Killing spinor can be chosen either as

$$\epsilon_2 = a1 + \bar{a}e_1$$

or as

$$\epsilon_2 = be_{12} - \bar{b}e_2$$

where a, b complex spacetime functions.

N	$Stab(\epsilon_1, \dots, \epsilon_N)$	ϵ
1	$\mathbb{C}$	$1 + e_1$
2	$\mathbb{C}$	$1 + e_1, i(1 - e_1)$
	$\{1\}$	$1 + e_1, e_2 - e_{12}$
$3, 4$	$\{1\}$	

$\epsilon_1 = 1 + e_1$, $\epsilon_2 = a1 + \bar{a}e_1$

The spacetime admits a parallel, null, vector field $X = \partial_v$

$$\nabla X = 0\,, \quad g(X, X) = 0$$

The spacetime is a pp-wave

$$ds^2 = du(dv + Vdu + w_r dx^r) + g_{rs}dx^r dx^s$$

The scalar fields ϕ are holomorphic, $W = \partial_j W = 0$ and

$$F^a_{1\bar{1}} = -i\mu^a$$

$\epsilon_1 = 1 + e_1$, $\epsilon_2 = -\bar{b}e_2 + be_{12}$

The spacetime admits three Killing vector fields X, Y, Z and a vector field W such that

$$[W, X] = [W, Y] = [W, Z] = 0$$

and

$$[X, Y] = cZ\ , \quad [X, Z] = -2cX\ , \quad [Y, Z] = 2cY$$

where c is a constant.
The spacetime metric is

$$ds^2 = 2|b|^2[ds^2(M_3) + dy^2]$$

where $W = \partial_y$

$$ds^2(M_3) = du(dv - c^2v^2du) + (dx - cvdu)^2$$

ie either AdS_3 for $c \neq 0$ or $\mathbb{R}^{2,1}$ for $c = 0$. Therefore, the spacetime is a domain wall with homogeneous sections AdS_3 or $\mathbb{R}^{2,1}$.
Moreover

$$F^a = \mu^a = 0$$

The scalars ϕ and b depend only on y, and satisfy appropriate flow equations.

$N = 3$ and $N = 4$ backgrounds

Start from the $N = 3$ case. The gauge group is used to find a representative for the normal to the 3 Killing spinors. Choose for example

$$\nu = ie_2 + ie_{12}$$

The Killing spinors are

$$\epsilon_r = f_{rs}\eta_s$$

where $(\eta_s) = (1 + e_1, i(1 - e_1), e_2 - e_{12})$ and $f = (f_{rs})$ an invertible 3×3 matrix of spacetime functions.
The KSE imply that the gauge connection is flat and the scalars are constant

$$F^a_{AB} = \mathcal{D}_A\phi^i = D_iW = \mu^a = 0$$

and

$$R_{AB,CD}\Gamma^{CD}\eta_r + 2e^K W\bar{W}\Gamma_{AB}\eta_r = 0$$

Since the above integrability condition takes values in $\mathfrak{spin}(3,1)$ and three linearly independent spinors have isotropy group $\{1\}$

$$R_{AB,CD} = -e^{K} W\bar{W}(g_{AC}g_{BD} - g_{BC}g_{AD})$$

and the spacetime is locally either $\mathbb{R}^{3,1}$ or AdS_4.

- All $N = 3$ backgrounds are locally maximally supersymmetric
- There are $N = 3$ backgrounds which arise from discrete identifications of maximally supersymmetric ones [J Figueroa O'Farrill, Gutowski, Sabra]
- The maximally supersymmetric backgrounds are locally isometric to either $\mathbb{R}^{3,1}$ or to AdS_4

Killing spinor equations

The Killing spinor equations of Heterotic supergravities are

$$\mathcal{D}\epsilon = \hat{\nabla}\epsilon = \nabla\epsilon + \frac{1}{2}H\epsilon + \mathcal{O}(\alpha') = 0\,, \quad \mathcal{F}\epsilon = F\epsilon + \mathcal{O}(\alpha') = 0\,,$$
$$\mathcal{A}\epsilon = d\Phi\epsilon - \frac{1}{2}H\epsilon + \mathcal{O}(\alpha') = 0$$

These are valid up to 2-loops in the sigma model calculation.
It is convenient to solve them in the order

gravitino → gaugino → dilatino

The gravitino and gaugino have a straightforward Lie algebra interpretation while the solution of the gaugino is more involved. All have been solved [Gran, Lohrmann, GP; hep-th/0510176], [Gran, Roest, Sloane, GP; hep-th/0703143].

$Spin(9,1)$ Spinors

$Spin(9,1)$ admits two inequivalent real chiral (Majorana-Weyl) representations Δ^+_{16}, Δ^-_{16}. They can be described in terms of forms as follows: Take $\mathbb{C}^5 = \mathbb{C} < e^1, \ldots, e^5 >$, equipped with the standard Hermitian inner product $< \cdot, \cdot >$.
The Dirac representation Δ_c is identified with the exterior algebra $\Lambda^*(\mathbb{C}^5)$ and the complex chiral representations are $\Delta^+_c = \Lambda^{\text{even}}(\mathbb{C}^5)$ and $\Delta^-_c = \Lambda^{\text{odd}}(\mathbb{C}^5)$.
In particular

$$\Gamma_0\eta = -e_5 \wedge \eta + e_5 \lrcorner \eta \,, \quad \Gamma_5 = e_5 \wedge \eta + e_5 \lrcorner \eta$$
$$\Gamma_i = e_i \wedge \eta + e_i \lrcorner \eta \,, \quad \Gamma_{i+5} = ie_i \wedge \eta - ie_i \lrcorner \eta$$

A reality condition can be constructed using the anti-linear map $R = -\Gamma_0 B*$, ie the real spinors are those that satisfy

$$\eta^* = \Gamma_{6789}\eta$$

The real and imaginary parts of 1 are

$$1 + e_{1234} \,, \quad i(1 - e_{1234})$$

The real spinors are multi-forms.

Gravitino

The gravitino Killing spinor equation is

$$\mathcal{D}\epsilon = \hat{\nabla}\epsilon = \nabla\epsilon + \frac{1}{2}H\epsilon = 0$$

where $\hat{\nabla}$ is a metric connection with skew-symmetric torsion H, and so for generic backgrounds

$$\mathrm{hol}(\hat{\nabla}) = G = Spin(9,1)$$

In addition

$$\hat{\nabla}\epsilon = 0 \Rightarrow \hat{R}\epsilon = 0$$

So either

$$\mathrm{Stab}(\epsilon) = \{1\} \Longrightarrow \hat{R} = 0$$

all spinors are parallel and M is parallelizable (group manifold if $dH = 0$) [Figueroa O'Farrill, Kawano, Yamaguchi] or

$$\mathrm{Stab}(\epsilon) \neq \{1\} \Longrightarrow \epsilon \text{ singlets}$$

$\mathrm{Stab}(\epsilon) \subset Spin(9,1)$ and $\mathrm{hol}(\hat{\nabla}) \subseteq \mathrm{Stab}(\epsilon)$.

Parallel spinors

L	$Stab(\epsilon_1, \dots, \epsilon_L)$	parallel ϵ
1	$Spin(7) \ltimes \mathbb{R}^8$	$1 + e_{1234}$
2	$SU(4) \ltimes \mathbb{R}^8$	1
3	$Sp(2) \ltimes \mathbb{R}^8$	$1,\ i(e_{12} + e_{34})$
4	$(SU(2) \times SU(2)) \ltimes \mathbb{R}^8$	$1,\ e_{12}$
5	$SU(2) \ltimes \mathbb{R}^8$	$1,\ e_{12},\ e_{13} + e_{24}$
6	$U(1) \ltimes \mathbb{R}^8$	$1,\ e_{12},\ e_{13}$
8	$\mathbb{R}^8$	$1,\ e_{12},\ e_{13},\ e_{14}$
2	G_2	$1 + e_{1234},\ e_{15} + e_{2345}$
4	$SU(3)$	$1,\ e_{15}$
8	$SU(2)$	$1,\ e_{12},\ e_{15},\ e_{25}$
16	$\{1\}$	$\Delta^+_{\mathbf{16}}$

- There are differences with the holonomy groups that appear in the Berger classification
- There are compact and non-compact isotropy groups which lead to geometries with different properties
- There is a restriction on the number of parallel spinors. This is a difference with the type II case
- The isotropy group of more than 8 spinors is $\{1\}$

- The table has been given constructed at various stages in [Acharya, Figueroa-O'Farrill, Spence, Stanciu], [Figueroa-O'Farrill] and [Gran, Roest, Sloane, GP].

Dilatino

The dilatino KSE is

$$d\Phi\zeta - \frac{1}{2}H\zeta = 0$$

Some of the solutions of the gravitino $\epsilon_1, \ldots, \epsilon_L$ may not solve the dilatino KSE. To choose the solutions $\zeta = \sum_r f_r \epsilon_r$ of the dilatino KSE use as gauge symmetry transformations,

$$\Sigma(\mathcal{P}) = \text{Stab}(\mathcal{P})/\text{Stab}(\epsilon_1, \ldots, \epsilon_L)$$

where $\mathcal{P}$ is the L-plane of spinors that solve both the gravitino, and $\text{Stab}(\mathcal{P})$ are those transformations of $Spin(9,1)$ that preserve $\mathcal{P}$.

- The gaugino KSE can be also solved using the $\Sigma(\mathcal{P})$ groups.
- If $N > L/2$, it is convenient to use $\Sigma(\mathcal{P})$ to choose the normals to the Killing spinors.

L	$Stab(\epsilon_1, \dots, \epsilon_L)$	$\Sigma(\mathcal{P})$
1	$Spin(7) \ltimes \mathbb{R}^8$	$Spin(1,1)$
2	$SU(4) \ltimes \mathbb{R}^8$	$Spin(1,1) \times U(1)$
3	$Sp(2) \ltimes \mathbb{R}^8$	$Spin(1,1) \times SU(2)$
4	$(SU(2) \times SU(2)) \ltimes \mathbb{R}^8$	$Spin(1,1) \times Sp(1) \times Sp(1)$
5	$SU(2) \ltimes \mathbb{R}^8$	$Spin(1,1) \times Sp(2)$
6	$U(1) \ltimes \mathbb{R}^8$	$Spin(1,1) \times SU(4)$
8	$\mathbb{R}^8$	$Spin(1,1) \times Spin(8)$
2	G_2	$Spin(2,1)$
4	$SU(3)$	$Spin(3,1) \times U(1)$
8	$SU(2)$	$Spin(5,1) \times SU(2)$
16	$\{1\}$	$Spin(9,1)$

- The $\Sigma(\mathcal{P})$ groups are a product of a *Spin* group and a R-symmetry group, reminiscent of lower-dimensional supergravities.

The list of all possible cases is as follows:

L	$\mathrm{Stab}(\epsilon_1, \dots, \epsilon_L)$	N
1	$Spin(7) \ltimes \mathbb{R}^8$	1(1)
2	$SU(4) \ltimes \mathbb{R}^8$	1(1), 2(1)
3	$Sp(2) \ltimes \mathbb{R}^8$	1(1), 2(1), 3(1)
4	$(\times^2 SU(2)) \ltimes \mathbb{R}^8$	1(1), 2(1), 3(1), 4(1)
5	$SU(2) \ltimes \mathbb{R}^8$	1(1), 2(1), 3(1), 4(1), 5(1)
6	$U(1) \ltimes \mathbb{R}^8$	1(1), 2(1), 3(1), 4(1), 5(1), 6(1)
8	$\mathbb{R}^8$	1(1), 2(1), 3(1), 4(1), 5(1), 6(1), 7(1), 8(1)
2	G_2	1(1), 2(1)
4	$SU(3)$	1(1), 2(2), 3(1), 4(1)
8	$SU(2)$	1(1), 2(2), 3(3), 4(6), 5(3), 6(2), 7(1), 8(1)
16	$\{1\}$	8(2), 10(1), 12(1), 14(1), 16(1)

- The cases noted in red are those for which all parallel spinors are Killing $N = L$, and the case in blue does not occur. In general $N \leq L$
- The number in parenthesis denotes the different geometries for a given N

Non-compact holonomy

The solutions of the KSE are characterized by the isotropy group of the parallel spinors $\mathrm{Stab}(\epsilon_1, \ldots, \epsilon_L)$ and the number N of solutions to the KSE.

Given $N \leq L$, $N \neq 7$, there is a case with $\tilde{L}$ parallel spinors such that $N = \tilde{L}$.

- The geometry of backgrounds with N Killing and L parallel, $N < L$, is a special case of those with $\tilde{L} = N$ parallel spinors
- The $N = 7$ case is treated differently.

Thus it suffices to consider those solutions for which all parallel spinors are Killing, $N \neq 7$.

Given $\mathrm{Stab}(\epsilon_1,\dots,\epsilon_L)=K\ltimes\mathbb{R}^8$ and $\mathrm{hol}(\hat\nabla)\subseteq K\ltimes\mathbb{R}^8$, such backgrounds admit $\hat\nabla$-parallel forms of the type

$$e^-\,,\quad e^-\wedge\phi$$

where e^- is a null 1-form and ϕ is a fundamental form of K.

$$\hat\nabla e^-=0 \Longleftrightarrow e_+\ \text{ Killing vector},\quad de^-=i_{e_+}H$$

where $e^-(Y)=g(e_+,Y)$, g metric.
Let I the trivial line bundle along e_+. Then

$$0\to I\to \mathrm{Ker}\,e^-\to\xi_{TM}\to 0$$

ξ_{TM} has rank 8 and is identified with the “transverse to the lightcone” directions in TM of the spacetime M. Similarly, the “transverse to the lightcone” forms can be defined.

$SU(4)\ltimes\mathbb{R}^8$, $L=2$

The $\hat{\nabla}$-parallel forms are

$$e^-\ ,\quad e^-\wedge\omega\ ,\quad e^-\wedge\chi$$

The metric and 3-form can be written as

$$\begin{aligned} ds^2 &= 2e^-e^+ + \delta_{ij}e^ie^j\ ,\quad i,j=1,\ldots,8 \\ H &= e^+\wedge de^- + \frac{1}{2}(h^{\mathfrak{su}(4)} + h^{\mathfrak{su}^\perp(4)})_{ij}e^-\wedge e^i\wedge e^j + \frac{1}{3!}\tilde{H}_{ijk}e^i\wedge e^j\wedge e^k \end{aligned}$$

and

$$\partial_+\Phi = 0\ ,\ \ 2\partial_i\Phi - H_{-+i} = (\tilde{\theta}_\omega)_i\ ,\ \ \tilde{H} = -i_{\tilde{I}}d\omega = -\star(\tilde{d}\omega\wedge\omega) - \frac{1}{2}\star(\tilde{\theta}_\omega\wedge\omega\wedge\omega)$$

subject to the geometric conditions

$$de^-\in\mathfrak{su}(4)\oplus_s\mathbb{R}^8\ ,\quad \tilde{\mathcal{N}}(I)=0\ ,\quad \tilde{\theta}_\omega = \tilde{\theta}_{\mathrm{Re}\,\chi}$$

where $\omega=\tilde{\omega}$, is the Hermitian form, $\tilde{\mathcal{N}}$ is the Nijenhuis tensor and

$$\tilde{\theta}_\omega = -\star(\star\tilde{d}\omega\wedge\omega)\ ,\quad \tilde{\theta}_{\mathrm{Re}\,\chi} = -\frac{1}{4}\star(\star\tilde{d}\mathrm{Re}\,\chi\wedge\mathrm{Re}\,\chi)$$

are Lee forms. The 2-form $h^{\mathfrak{su}(4)}\in\mathfrak{su}(4)$ is not determined by the KSE.
The expression for $\tilde{H}$ is as that for 8-manifolds with $SU(4)$ structure and compatible connection with skew-symmetric torsion [Friedrich, Ivanov].

$N \geq 3$

If the isotropy group is $K \ltimes \mathbb{R}^8$, the metric and 3-form can be written as

$$\begin{aligned} ds^2 &= 2e^- e^+ + \delta_{ij} e^i e^j \;, \quad i,j = 1,\ldots,8 \\ H &= e^+ \wedge de^- + \frac{1}{2}(h^{\mathfrak{k}} + h^{\mathfrak{k}\perp})_{ij} e^- \wedge e^i \wedge e^j + \frac{1}{3!}\tilde{H}_{ijk} e^i \wedge e^j \wedge e^k \end{aligned}$$

and

$$\partial_+ \Phi = 0 \;,\; 2\partial_i \Phi - H_{-+i} = (\tilde{\theta}_r)_i \;,\; \tilde{H} = -i_{\tilde{I}_r} \tilde{d}\omega_r = -\star(\tilde{d}\omega_r \wedge \omega_r) - \frac{1}{2}\star(\tilde{\theta}_r \wedge \omega_r \wedge \omega_r)$$

subject to the geometric conditions

$$de^- \in \mathfrak{k} \oplus_s \mathbb{R}^8 \;, \quad \tilde{\mathcal{N}}(I_r) = 0 \;, \quad i_{\tilde{I}_r} \tilde{d}\omega_r = i_{\tilde{I}_s} \tilde{d}\omega_s \;, \quad \tilde{\theta}_r = \tilde{\theta}_s \;, \quad r \neq s$$

where ω_r and θ_r are the Hermitian and Lee forms. The component $h^{\mathfrak{k}}$ is not determined by the field equations.

In addition, the endomorphisms I_r of ξ_{TM} associated with ω_r satisfy the Clifford relation as

N	$\mathrm{S}tab(\epsilon_1, \dots, \epsilon_L)$	Clifford
2	$SU(4) \ltimes \mathbb{R}^8$	$\mathrm{Cliff}(\mathbb{R})$
3	$Sp(2) \ltimes \mathbb{R}^8$	$\mathrm{Cliff}(\mathbb{R}^2)$
4	$(SU(2) \times SU(2)) \ltimes \mathbb{R}^8$	$\mathrm{Cliff}(\mathbb{R}^3)$
5	$SU(2) \ltimes \mathbb{R}^8$	$\mathrm{Cliff}(\mathbb{R}^4)$
6	$U(1) \ltimes \mathbb{R}^8$	$\mathrm{Cliff}(\mathbb{R}^5)$
7	$\mathbb{R}^8$	$\mathrm{Cliff}(\mathbb{R}^6)$
8	$\mathbb{R}^8$	$\mathrm{Cliff}(\mathbb{R}^7)$

In the $N = 8$, $\mathbb{R}^8$ case, $\tilde{H} = 0$ and $e^- \wedge de^- = 0$.

Holonomy Reduction

Consider $SU(4) \ltimes \mathbb{R}^8$. Since $\mathrm{hol}(\hat{\nabla}) \subseteq SU(4) \ltimes \mathbb{R}^8$, the expected $\hat{\nabla}$-parallel forms are

$$e^- \,, \quad e^- \wedge \omega_I \,, \quad e^- \wedge \mathrm{Re}\,\chi \,, \quad e^- \wedge \mathrm{Im}\,\chi$$

However, the field equations,

$$dH = 0 \,, \quad \mathrm{hol}(\hat{\nabla}) \subseteq SU(4) \ltimes \mathbb{R}^8$$

imply that

$$\tau_1 = H_{+ij}\omega_I^{ij} e^+ \,, \\ \tau_2 = \mathcal{N} \,, \quad \tau_3 = 2d\Phi - \theta_{\omega_I} \,,$$

which do not vanish for $N = 1$, are ALSO $\hat{\nabla}$-parallel. Similarly for the other $K \ltimes \mathbb{R}^8$ cases. The consequences are that

- The existence of $N < L$ supersymmetric backgrounds requires that $\mathrm{hol}(\hat{\nabla}) \subset \mathrm{Stab}(\epsilon)$.
- If $\mathrm{hol}(\hat{\nabla}) = \mathrm{Stab}(\epsilon)$, then the gravitino KSE implies the dilatino one and ALL parallel are Killing $L = N$, *i.e.* there are no $N < L$ backgrounds

Compact holonomy

The $\hat{\nabla}$-parallel forms in this case are

$$e^a\,, \quad \phi$$

where e^a are $\hat{\nabla}$-parallel 1-forms and ϕ are the fundamental forms of $\mathrm{hol}(\hat{\nabla}) \subseteq \mathrm{Stab}(\epsilon_1, \ldots, \epsilon_L)$.

$$\hat{\nabla} e^a = 0 \Longleftrightarrow e_a \;\; \text{Killing vector}, \quad de^a = i_{e_a} H$$

where $e^a(Y) = g(e_a, Y)$, g metric.

$Stab(\epsilon_1, \ldots, \epsilon_L)$	$1-$ forms	$\mathfrak{h}$
G_2	≥ 3	$\mathbb{R}^3, \mathfrak{sl}(2,\mathbb{R})$
$SU(3)$	≥ 4	$\mathbb{R}^4, \mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R}, \mathfrak{su}(2) \oplus \mathbb{R}, \mathfrak{cw}_4$
$SU(2)$	≥ 6	$\mathbb{R}, \mathfrak{sl}(2,\mathbb{R}), \mathfrak{su}(2), \mathfrak{cw}_4, \mathfrak{cw}_6$

- The second column denotes the minimal number of $\hat{\nabla}$-parallel 1-forms
- The third column denotes the Lorentzian Lie algebra, $\mathfrak{h}$, of the associated vector fields under Lie brackets

There is no such a straightforward relation between the cases where all parallel spinors are Killing and the rest.

L	$\mathrm{Stab}(\epsilon_1, \dots, \epsilon_L)$	N
1	$Spin(7) \ltimes \mathbb{R}^8$	1(1)
2	G_2	1(1), 2(1)
2	$SU(4) \ltimes \mathbb{R}^8$	1(1), 2(1)
4	$SU(3)$	1(1), 2(2), 3(1), 4(1)

The $N = 3$, $SU(3)$ case does not have a direct relation to those for which $N = L$.
There are several $SU(2)$ cases with this property.
To describe the geometry, consider some cases for $N = L$.

G_2

Consider $\mathrm{hol}(\hat{\nabla})=G_2$ and $N=L=2$, $\mathfrak{h}=\mathbb{R}^3,\mathfrak{sl}(2,\mathbb{R})$. The spacetime $M=P(H,B;\pi)$, Lie $H=\mathfrak{h}$ equipped with a connection $\lambda=e$. Then

$$ds^2=\eta_{ab}\lambda^a\lambda^b+\pi^*d\tilde{s}^2$$
$$H=\frac{1}{3}\eta_{ab}\lambda^a\wedge d\lambda^b+\frac{2}{3}\eta_{ab}\lambda^a\wedge\mathcal{F}^b+\pi^*\tilde{H}$$

where $(d\tilde{s}^2,\tilde{H})$ on B are data compatible with a connection with shew-symmetric torsion $\hat{\tilde{\nabla}}$ on B such that $\mathrm{hol}(\hat{\tilde{\nabla}})=G_2$,

$$\tilde{\theta}_\varphi=2\tilde{d}\Phi\ ,\quad \partial_a\Phi=0\ ,$$

and λ G_2-instanton connection. In particular, on B [Friedrich, Ivanov]

$$\tilde{H}=-\frac{r}{6}(d\varphi,\star\varphi)\varphi+\star d\varphi+\star(\tilde{\theta}_\varphi\wedge\varphi)$$
$$\tilde{d}\star\varphi=-\tilde{\theta}_\varphi\wedge\star\varphi$$

$r=0$ if λ abelian, and $r=1$ if λ non-abelian, where

$$\tilde{\theta}_\varphi=\star(\star\tilde{d}\varphi\wedge\varphi)$$

is the Lee form of the fundamental G_2 form φ. B is conformally co-symplectic.

$SU(2)$

Consider $\mathrm{hol}(\hat{\nabla}) \subseteq SU(2)$ and $N = L = 8$. First $\mathfrak{h}$ is a self-dual Lorentzian Lie algebra

$$\mathbb{R}^{5,1}, \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{su}(2), \mathfrak{cw}_6$$

The spacetime $M = P(H, B; \pi)$, Lie $H = \mathfrak{h}$ equipped with a connection $\lambda = e$. Then

$$ds^2 = \eta_{ab}\lambda^a\lambda^b + \pi^* d\tilde{s}^2$$
$$H = \frac{1}{3}\eta_{ab}\lambda^a \wedge d\lambda^b + \frac{2}{3}\eta_{ab}\lambda^a \wedge \mathcal{F}^b + \pi^*\tilde{H}$$

where $(d\tilde{s}^2, \tilde{H})$ on B are data compatible with a connection with skew-symmetric torsion $\hat{\tilde{\nabla}}$ on B such that $\mathrm{hol}(\hat{\tilde{\nabla}}) \subseteq SU(2)$, ie B is an HKT manifold,

$$\tilde{\theta}_{\omega_1} = 2\tilde{d}\Phi\ , \quad \partial_a\Phi = 0\ ,$$

and λ an instanton connection on B.
Since B is conformally balanced, then B is conformal to a hyper-Kähler, and

$$\tilde{H} = -\star_{\mathrm{hk}}\, df\ , \quad e^{2\Phi} = f\ , \quad d\tilde{s}^2 = f ds^2_{\mathrm{hk}}\ .$$

Moreover

$$dH = \eta_{ab}\mathcal{F}^a \wedge \mathcal{F}^b + d\tilde{H}\ .$$

Some solutions

Consider the case that $dH = 0$.

If P is trivial, then one class of solutions is

$$\mathbb{R}^{5,1} \times B_{\text{hk}} \,, \quad AdS_3 \times S^3 \times B_{\text{hk}} \,, \quad CW_6 \times B_{\text{hk}}$$

All these solutions have constant dilaton.

Another solution is the heterotic 5-brane (allowing for delta-function sources) [Callan, Harvey, Strominger]

$$ds^2 = ds^2(\mathbb{R}^{5,1}) + f ds^2(\mathbb{R}^4) \,, \quad e^{2\Phi} = f \,, \quad H = -\star_{\mathbb{R}^4} df \,, \quad f = 1 + \frac{Q}{|x|^2} \,, \quad B_{\text{hk}} = \mathbb{R}^4$$

There are two asymptotic regions.

- The asymptotic infinity $|x| \to \infty$.
  The metric approaches Minkowski spacetime $ds^2(\mathbb{R}^{9,1})$.
- The near horizon limit $|x| \to 0$. The metric approaches $ds^2(\mathbb{R}^{5,1}) + ds^2(S^3) + ds^2(\mathbb{R})$, the dilaton is linear and $H = d\mathrm{vol}(S^3)$.

New solutions

Suppose $B_{\rm hk} = \mathbb{R}^4$ and λ is an $SU(2)$ self-dual connection on $B_{\rm hk}$. Such connections can be constructed using the t'Hooft ansatz or ADHM.
The solution for a one instanton connection is

$$ds^2 = ds^2(AdS_3) + \delta_{ab}\lambda^a\lambda^b + f ds^2(\mathbb{R}^4)\ , \quad e^{2\Phi} = f\ , \quad f = 1 + 4\frac{|x|^2 + 2\rho^2}{(|x|^2 + \rho^2)^2}$$

There is one asymptotic region as $|x| \to \infty$ where the metric approaches $ds^2(AdS_3) + ds^2(S^3) + ds^2(\mathbb{R}^4)$ and the dilaton is constant.

The geometry near $|x| \to 0$ is again $ds^2(AdS_3) + ds^2(S^3) + ds^2(\mathbb{R}^4)$ and the dilaton is constant. But $|x| = 0$ is not an asymptotic point.

The solution is smooth.

More solutions can be constructed by taking multi-instanton connections.

$\mathbb{R}^8$

The conditions that arise from the Killing spinor equations are

$$\text{hol}(\hat{\nabla}) \subseteq \mathbb{R}^8\,, \quad \partial_+\Phi = 0\,, \quad e^- \wedge de^- = 0\,, \quad H_{ijk} = 0\,, \quad 2\partial_i\Phi - H_{-+i} = 0\,.$$

Takings $e_+ = \partial_u$ and $e^- = f^{-1}(y,v)dv$, the fields can be written as

$$\begin{aligned} ds^2 &= 2f^{-1}dv(du + Vdv + n_I dy^I) + \delta_{IJ}dy^I dy^J \\ H &= d(e^- \wedge e^+) \\ e^{2\Phi} &= f^{-1}(v,y)g(v) \end{aligned}$$

where $e^+ = du + Vdv + n_I dy^I$.
These solutions have the interpretation of either a fundamental string, or a pp-wave, and/or their superpositions which may include a null rotation.

Conclusions

- The Killing spinor equations of $\mathcal{N} = 1$ $D = 4$ supergravity have been solved in ALL cases. Solutions include pp-waves, and domain walls with homogenous sections $\mathbb{R}^{3,1}$ and AdS_3.
- The Killing spinor equations of heterotic supergravity have been solved in ALL cases, and the conditions on the geometry of the spacetime have been determined.
- If the isotropy group of the parallel spinors is non-compact, $K \ltimes \mathbb{R}^8$, $K = Spin(7), SU(4), \times^2 SU(2), SU(2), U(1), \{1\}$, then the spacetime admits a null $\hat{\nabla}$-parallel 1-form, and certain compatible K-structure on the transverse 8-directions to the lightcone.
- If the isotropy group of the parallel spinors is compact, $K = G_2, SU(3), SU(2), \{1\}$, then in some cases the spacetime M is a principal bundle with either an abelian or non-abelian fibre equipped with a connection. The base space admits an appropriate compatible K-type of structure. There are new solutions with 8 Killing spinors.