$\mathbf{GL}(2, \mathbb{R})$ geometry of ODEs

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and has 3 *distinct* principal curvatures iff $S = \mathbf{S}^{n-1} \cap P_c$, where

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and $w = w(a)$ is a homogeneous 3rd order *polynomial* in variables (a^i) such that

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- if $n = 5$ the tensor Υ is given by:

$$\Upsilon_{ijk} a_i a_j a_k = w(a) = \det \begin{pmatrix} a_5 - \sqrt{3} a_4 & \sqrt{3} a_3 & \sqrt{3} a_2 \\ \sqrt{3} a_3 & a_5 + \sqrt{3} a_4 & \sqrt{3} a_1 \\ \sqrt{3} a_2 & \sqrt{3} a_1 & -2 a_5 \end{pmatrix}$$

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- if $n = 5$, 8, 14 and 26 we take:

$$w(a) = \det \begin{pmatrix} a_5 - \sqrt{3}a_4 & \sqrt{3}\alpha_3 & \sqrt{3}\alpha_2 \\ \sqrt{3}\overline{\alpha}_3 & a_5 + \sqrt{3}a_4 & \sqrt{3}\alpha_1 \\ \sqrt{3}\overline{\alpha}_2 & \sqrt{3}\overline{\alpha}_1 & -2a_5 \end{pmatrix}$$

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where for $n = 5$:

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where for $n = 8$:

$$\alpha_1 = a_1 + a_6 \mathrm{i}$$
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  where for $n = 14$:

$$\alpha_1 = a_1 + a_6 \mathrm{i} + a_9 \mathrm{j} + a_{10} \mathrm{k}$$
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$$\alpha_2 = a_2 + a_7 \mathrm{i} + a_{11} \mathrm{j} + a_{12} \mathrm{k} + a_{19} \mathrm{p} + a_{20} \mathrm{q} + a_{21} \mathrm{r} + a_{22} \mathrm{s},$$
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- $n = 6$-dimensional compact Riemannian manifold (M, g) which, in addition to the Levi-Civita connection ∇^{LC}, is equipped with:
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- *special* Riemannian structure $(M, g, \nabla^{T}, T, \Psi)$ should satisfy a number of field equations including:

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Question: How to construct solutions to the above equations in n dimensions?

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- Among the irreducible $\mathbf{SO}(3)$ geometries in dimension 5 we distinguished the *nearly integrable* ones, for which the tensor Υ is a Killing tensor for the Levi-Civita connection:

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- S. Chiossi + A. Fino found plenty of examples of such structures possessing 5-dimensional symmetry groups.

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- Coefficients a_i of a 4th order polynomial

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form a carrier space for the 5-dimensional irreducible representation of the $\mathbf{GL}(2,\mathbb{R})$ group; this is induced on $\mathbb{R}^5$ by the defining action of $\mathbf{GL}(2,\mathbb{R})$ on $(x,y) \in \mathbb{R}^2$.

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- A polynomial I, in variables a_i, is called an *algebraic invariant* of $w_4(x,y)$ if it changes according to

$$I \rightarrow I' = (\det b)^p \ I, \qquad b \in \mathbf{GL}(2,\mathbb{R})$$

  under the action of this 5-dimensional representation on a_is.

- The lowest order invariants of $w_4(x, y)$ are:

$$I_2 = 3a_2^2 - 4a_1a_3 + a_0a_4$$

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- Defining Υ_{ijk} and g_{ij} via

$$\Upsilon_{ijk} a_i a_j a_k = 3\sqrt{3} I_3$$

$$g_{ij} a_i a_j = I_2,$$

one can check that the so defined g_{ij} and Υ_{ijk} satisfy the desidered relations i)-iii).

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  ⋆ g is a $(3,2)$ signature metric; Υ is a rank three totally symmetric tensor
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  ⋆ $(g,\Upsilon)\ \sim\ (g',\Upsilon')\ \Leftrightarrow\ g'=\mathrm{e}^{2\phi}g,\quad \Upsilon'=\mathrm{e}^{3\phi}\Upsilon.$

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  ⋆ g is a $(3,2)$ signature metric; Υ is a rank three totally symmetric tensor
  ⋆ $g^{ij}\Upsilon_{ijk} = 0,$
  ⋆ $g^{ab}(\Upsilon_{jka}\Upsilon_{lmb} + \Upsilon_{lja}\Upsilon_{kmb} + \Upsilon_{kla}\Upsilon_{jmb}) = g_{jk}g_{lm} + g_{lj}g_{km} + g_{kl}g_{jm},$
  ⋆ $(g,\Upsilon) \ \sim\ (g',\Upsilon') \ \Leftrightarrow\ g' = \mathrm{e}^{2\phi}g, \quad \Upsilon' = \mathrm{e}^{3\phi}\Upsilon.$

- The stabilizer of the conformal class $[(g,\Upsilon)]$ is the irreducible $\mathbf{GL}(2,\mathbb{R})$ in dimension five.

Irreducible $\mathbf{GL}(2, \mathbb{R})$ geometry in dimension 5

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A 5-dimensional manifold M^5 equipped with a class of triples $[(g,\Upsilon,A)]$ such that:

- g is a $(3,2)$ signature metric; Υ is a rank three totally symmetric traceless tensor field; A is a 1-form on M^5
- $g^{ab}(\Upsilon_{jka}\Upsilon_{lmb}+\Upsilon_{lja}\Upsilon_{kmb}+\Upsilon_{kla}\Upsilon_{jmb})=g_{jk}g_{lm}+g_{lj}g_{km}+g_{kl}g_{jm}$,
- $(g,\Upsilon,A)\sim(g',\Upsilon',A')\ \Leftrightarrow\ \Big(g'=\mathrm{e}^{2\phi}g,\ \Upsilon'=\mathrm{e}^{3\phi}\Upsilon,\ A'=A-2\mathrm{d}\phi\Big)$,

is called an *irreducible* $\mathbf{GL}(2,\mathbb{R})$ structure in dimension five.

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$$\overset{W}{\nabla}_X g + A(X)g = 0.$$

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$$\overset{W}{\nabla}_X g + A(X)g = 0.$$

- An irreducible $\mathbf{GL}(2, \mathbb{R})$ structure $(M^5, [(g, \Upsilon, A)])$ is called *nearly integrable* iff tensor Υ is a *conformal* Killing tensor for $\overset{W}{\nabla}$:

$$\overset{W}{\nabla}_X \Upsilon(X, X, X) + \tfrac{1}{2} A(X) \Upsilon(X, X, X) = 0, \qquad \forall X \in \mathrm{T}M^5.$$

Characteristic connection

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- To achieve the uniqueness one requires the that torsion T of ∇, considered as an element of $\bigotimes^3 \mathrm{T}^* M^5$, seats in a 10-dimensional subspace $\bigwedge^3 \mathrm{T}^* M^5$.

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- Can we produce examples of the nearly integrable $\mathbf{GL}(2,\mathbb{R})$ geometries in dimension five? Can we produce examples with 'pure' torsion in $\bigwedge_3$ or $\bigwedge_7$? Can we produce nonhomogeneous examples?

A well known fact

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- Ordinary differential equation $y^{(5)} = 0$ has $\mathbf{GL}(2, \mathbb{R}) \times_\rho \mathbb{R}^5$ as its group of contact symmetries. Here $\rho : \mathbf{GL}(2, \mathbb{R}) \to \mathbf{GL}(5, \mathbb{R})$ is the 5-dimensional irreducible representation of $\mathbf{GL}(2, \mathbb{R})$.

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- This, in particular, means that the solution space M^5 of this ODE, which is $\mathbb{R}^5 \ni (a_0, a_1, a_2, a_3, a_4)$ of $y = a_0 + 4a_1 x + 6a_2 x^2 + 4a_3 x^3 + a_4 x^4$, is a *'conformal symmetric space'*

$$\Big(\mathbf{GL}(2,\mathbb{R}) \times_\rho \mathbb{R}^5\Big)/\mathbf{GL}(2,\mathbb{R}) = M^5,$$

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- What about more complicated 5th order ODEs?

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$$150F_4DF_3 + 200F_2F_4 - 140F_4^2DF_4 + 130F_3F_4^2 + 14F_4^4 = 0$$

$$1250D^2F_2 - 6250DF_1 + 1750DF_3DF_4 - 2750F_2DF_4 -$$

$$875F_3DF_3 + 1250F_2F_3 - 500F_4DF_2 + 700(DF_4)^2F_4 +$$

$$1250F_1F_4 - 1050F_3F_4DF_4 + 350F_3^2F_4 - 350F_4^2DF_3 +$$

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- Then the 5-dimensional solution space of the equation is naturally equipped with a nearly integrable $\mathbf{GL}(2,\mathbb{R})$ structure.

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- Every nearly integrable $\mathbf{GL}(2,\mathbb{R})$ structure obtained in this way has torsion of its characteristic connection of the 'pure' type $T \in \bigwedge_3$.

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- We call the three conditions on F the Wünschmann-like conditions.

Examples of F satisfying the Wünschmann-like conditions

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- The three differential equations

$$y^{(5)} = c\Big(\frac{5y^{(3)3}(5-27cy''^2)}{9(1+cy''^2)^2} + 10\frac{y''y^{(3)}y^{(4)}}{1+cy''^2}\Big),$$

with $c = +1, 0, -1,$ represent the only three contact nonequivalent classes of Wünschmann-like ODEs having the corresponding nearly integrable $\mathbf{GL}(2,\mathbb{R})$ structures $(M^5, [g, \Upsilon, A])$ with the characteristic connection with vanishing torsion.

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- If $c = 0$ we have $y^{(5)} = 0$ and the corresponding $\mathbf{GL}(2,\mathbb{R})$ structure on the solution space M^5 is flat.

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- Similarly in $c = -1$ case the manifold M^5 can *locally* be identified with the conformal symmetric space $\mathbf{SL}(3,\mathbb{R})/\mathbf{SL}(2,\mathbb{R})$ with an Einstein g descending from the Killing form on $\mathbf{SL}(3,\mathbb{R})$.

- In all the three cases the holonomy of the Weyl connection $\overset{W}{\nabla}$ of structures $(M^5,[g,\Upsilon,A])$ is reduced to the $\mathbf{GL}(2,\mathbb{R})$. Thus $\overset{W}{\nabla}=\nabla$. For all the three cases the Maxwell 2-form $\mathrm{d}A\equiv 0$.

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- In both cases with $c\neq 0$ the metric g is not conformally flat.

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$$F=\frac{5(8y_3^3-12y_2y_3y_4+3y_1y_4^2)}{6(2y_1y_3-3y_2^2)},$$

$$F=\frac{5y_4^2}{3y_3}\pm y_3^{5/3},$$

represent four nonequivalent nearly integrable $\mathbf{GL}(2,\mathbb{R})$ structures corresponding to the different signs in the second expression and to the different signs of the denominator in the first expression. These structures have 6-dimensional symmety group and $\mathrm{d}A=0$.

$$F = \frac{1}{9(y_1^2 + y_2)^2} \times$$

$$\Big(5w\big(y_1^6 + 3y_1^4 y_2 + 9y_1^2 y_2^2 - 9y_2^3 - 4y_1^3 y_3 + 12y_1 y_2 y_3 + 4y_3^2 - 3y_4(y_1^2 + y_2)\big) +$$

$$45y_4(y_1^2 + y_2)(2y_1 y_2 + y_3) - 4y_1^9 - 18y_1^7 y_2 - 54y_1^5 y_2^2 - 90y_1^3 y_2^3 + 270y_1 y_2^4 +$$

$$15y_1^6 y_3 + 45y_1^4 y_2 y_3 - 405y_1^2 y_2^2 y_3 + 45y_2^3 y_3 + 60y_1^3 y_3^2 - 180y_1 y_2 y_3^2 - 40y_3^3\Big),$$

where

$$w^2 = y_1^6 + 3y_1^4 y_2 + 9y_1^2 y_2^2 - 9y_2^3 - 4y_1^3 y_3 + 12y_1 y_2 y_3 + 4y_3^2 - 3y_1^2 y_4 - 3y_2 y_4.$$

This again has 6-dimensional symmetry group, but now $F = \mathrm{d}A \neq 0$.

Nonhomogeneous example

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reduces Wünschmann-like conditions to a single ODE

$$90z^{4/3}(3q - 4z^{2/3})\frac{\mathrm{d}^2 q}{\mathrm{d}z^2} - 54z^{4/3}(\frac{\mathrm{d}q}{\mathrm{d}z})^2 + 30z^{1/3}(6q - 5z^{2/3})\frac{\mathrm{d}q}{\mathrm{d}z} - 25q = 0,$$

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in which $z = \frac{y_4^3}{y_3^4}$.
This equation may be solved explicitely giving example of ODEs having its nearly integrable structure being nonhomogeneous.

What about other orders of ODEs?

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- If a 3rd order ODE $y''' = F(x, y, y', y'')$ satisfies the Wünschmann condition

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- This conformal structure in dimension *three* is related to the quadratic $\mathbf{GL}(2,\mathbb{R})$ invariant $\Delta = a_0a_2 - a_1^2$ of $w_2(x,y) = a_0x^2 + 2a_1xy + a_2y^2$.

- If a 4th order ODE $y^{(4)} = F(x, y, y', y'', y''')$ satisfies the Wünschmann-like conditions

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$$80DF_2F_3 + 160F_1F_3 - 72DF_3F_3^2 + 88F_2F_3^2 + 9F_3^4 + 16000F_y = 0,$$

$$D = \partial_x + y_1\partial_y + y_2\partial_{y_1} + y_3\partial_{y_2} + F\partial_{y_3},$$

then it defines an irreducible $\mathbf{GL}(2, \mathbb{R})$ structure on the 4-dimensional space M^4 of its solutions.

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$$I_4 = -3a_1^2a_2^2 + 4_0a_2^3 + 4a_1^3a_3 - 6a_0a_1a_2a_3 + a_0^2a_3^2,$$

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This is a report on a *joint* work with my student Michał Godliński.