# The Time Slice Axiom in Perturbative Quantum Field Theory

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# Introduction

Principle of determinism: Given initial data, the laws of physics determine the evolution of a system for all times.
Valid in

- Classical Mechanics (Newton's Law)
- Quantum Mechanics (Schrödinger equation)
- Classical Electrodynamics (Maxwell's Equation)
- General relativity (somewhat involved consequence of Einstein's equation)

Does it hold also in quantum field theory?
Free fields: Heisenberg's equation coincides with the classical field equation. Thus the well-posedness of the Cauchy problem for the classical field equation implies the determinism of the quantum field.

Problems for interacting fields:

- Field equation has to be renormalized.
- Time zero fields i.g. ill defined.

Reformulation as Time Slice Axiom:

The algebra of observables generated by fields within an arbitrary small time slice coincides with the algebra of all observables.

(Haag-Schroer 1962 primitive causality).

# The time slice axiom for a free scalar field

$M \sim \Sigma \times \mathbb{R}$ globally hyperbolic spacetime with Cauchy surfaces $\Sigma \times \{t\}$, $t \in \mathbb{R}$
The algebra $\mathfrak{F}_0(M)$ of the free scalar field is the unital *-algebra generated by elements $\varphi(f)$, $f \in \mathcal{D}(M)$ with the relations

$$\varphi(\lambda f + g) = \lambda\varphi(f) + \varphi(g)\ ,\ \lambda \in \mathbb{C}, f, g \in \mathcal{D}(M)$$

$$\varphi(f)^* = \varphi(\overline{f})$$

$$[\varphi(f), \varphi(g)] = i\langle f, \Delta g\rangle$$

$$\varphi(Pf) = 0$$

with $\Delta = \Delta_R - \Delta_A$, $\Delta_R, \Delta_A$ retarded/advanced Green's functions for the Klein-Gordon operator $P = \Box + m^2$.

Time slice axiom:

Let $N$ be a neighbourhood of a Cauchy surface $\Sigma$ and let $f \in \mathcal{D}(M)$ with supp$f \subset J_+(\Sigma)$.

We have to show that there exist $g, h \in \mathcal{D}(M)$ with supp$g \subset N$ and $f = g + Ph$.

Let $\chi \in \mathcal{C}^\infty(M)$ with

$$\begin{aligned} \text{supp}(d\chi) &\subset N, \\ \chi \equiv 1 &\text{ on } J_+(\Sigma) \setminus N, \\ \chi \equiv 0 &\text{ on } J_-(\Sigma) \setminus N. \end{aligned}$$

Then supp$(f - P\chi\Delta_R f) \subset N$,

hence $g = f - P\chi\Delta_R f$ and $h = \chi\Delta_R f$ solve the problem.

(Dimock 1980, Fulling,Narcovich,Wald 1981).

(Analogously for supp$f \subset J_-(\Sigma)$)

# The algebra of Wick polynomials

Problem: $\mathfrak{F}_0(M)$ does not contain Wick products.

Garding-Wightman: Wick products are defined in a Hilbert space representation as operator valued distributions on a dense domain.

$$:\varphi(x_1)\cdots\varphi(x_n):=\frac{\delta^n}{\delta f(x_1)\cdots\delta f(x_n)}e^{\varphi(f)}e^{\frac{1}{2}\langle f,\omega_2 f\rangle}|_{f=0}$$

($\omega_2$ 2-point function of a Hadamard state $\omega$, valid in the GNS representation induced by $\omega$ on the minimal invariant domain containing the cyclic vector corresponding to $\omega$.)

Disadvantage: Dependence on the representation and on the choice of the dense domain.

New method
(Brunetti-Dütsch-Fredenhagen-Hollands-Köhler-Radzikowski-Wald 1995-2003)

$$\mathfrak{F}_0(M) = \mathfrak{F}_1(M)/\mathfrak{I}_1(M)$$

$\mathfrak{F}_1(M)$ is the algebra of polynomials on $\mathcal{C}^\infty(M)$ with the product

$$(F \star G)(\varphi) = \sum_n \frac{\hbar^n}{2^n n!} \langle F^{(n)}(\varphi), \omega_2^{\otimes n} G^{(n)}(\varphi) \rangle$$

$F^{(n)} = \frac{\delta^n F}{\delta \varphi^n}$ $n$th functional derivative,

$$\langle F^{(n)}(\varphi), f^{\otimes n} \rangle = \frac{d^n}{d\lambda^n} F(\varphi + \lambda f)|_{\lambda=0} \ .$$

$\mathfrak{I}_1(M)$ is the ideal (with respect to $\star$ or, equivalently, the pointwise product) generated by $\varphi(Pf)$.

Extension of $\mathfrak{F}_1(M)$ to polynomials with distributional functional derivatives whose wavefront sets satisfy

$$\mathrm{WF}(F^{(n)}) \cap (\overline{V_+}^n \cup \overline{V_-}^n) = \emptyset \ .$$

Properties of the extended algebra $\mathfrak{F}_2(M)$:

1. $\star$-product well defined because of wave front set of $\omega_2$.
2. $\mathfrak{F}_1(M)$ sequentially dense in $\mathfrak{F}_2(M)$ with respect to Hörmander topology
3. $\mathfrak{F}_2(M)$ contains smeared polynomials $\int dx \varphi(x)^n f(x)$, $f \in \mathcal{D}(M)$
4. Independence (up to isomorphy) of the choice of the Hadamard state
5. $\mathfrak{F}(M) := \mathfrak{F}_2(M)/\overline{\mathfrak{I}_1(M)}$ isomorphic to an algebra of Hilbert space operators in the GNS representation of the Hadamard state.

# Time slice axiom for Wick polynomials

Proof analogous to the case of the free field:

Method: Application of Hörmander's Theorem on the propagation of singularities

Implication: Given a distribution $f$ in $n$ variables which satisfies the condition on the wave front set and with supp$f \subset K^n$ with a compact region $K \subset J_+(\Sigma)$ for some Cauchy surface $\Sigma$. Let $N$ be a neigbourhood of $\Sigma$. Then there exists $g$ with supp$g \subset N^n$ and $h \in \sum \mathrm{image}(P_i)$ such that $f = g + h$, and $g, h$ satisfy the condition on the wave front set.

Time slice axiom: $\mathfrak{F}_2(M) = \mathfrak{F}_2(N) \text{ modulo } \overline{\mathfrak{I}_1(M)}$ if $N \subset M$ contains a Cauchy surface of $M$.

# Time slice axiom for interacting field theories

Interacting theories are constructed via time ordered products of Wick polynomials.

Generating functional: Formal $S$-matrix (time ordered exponential) as a map $S : \mathcal{D}(M, V) \to \mathfrak{F}(M)[[\hbar]]$
($V$ space of possible interactions).

Functional relation: (Bogoliubov)

$$S(f + g + h) = S(f + g) \star S(g)^{-1} \star S(g + h)$$

if supp$f$ is later than supp$h$.

Holds also true for the relative S-matrices (generating functionals of time ordered products of interacting fields with respect to coupling constant $k \in \mathcal{D}(M, V)$)

$$S_k(f) := S(k)^{-1} \star S(k + f)$$

Algebra of free fields:
$\mathfrak{A}_0(\mathcal{O})$ generated by $S(f)$, supp$f \subset \mathcal{O}$.

Algebra of interacting fields :
$\mathfrak{A}_k(\mathcal{O})$ generated by relative S-matrices $S_k(f)$, supp$f \subset \mathcal{O}$.

Adiabatic limit: Couplings $k \in \mathcal{C}^\infty(M)$

Crucial fact: The theory inside a region with compact closure does not depend on the behaviour of $k \in \mathcal{D}(M)$ outside of this region,

$$S_k(f) \to S_{k'}(f) \ , \ \text{supp} f \subset \mathcal{O}$$

is an isomorphism.
($\mathcal{O}$ causally convex globally hyperbolic subregion with compact closure, $k \equiv k'$ on a neighbourhood of the closure of $\mathcal{O}$).

Reason: $k' - k = a_+ + a_-$, $\mathrm{supp} a_\pm \cap J_\mp(\mathcal{O}) = \emptyset$, hence

$$\begin{aligned} S_{k'}(f) &= S(k')^{-1} S(k' + f) \\ &= S(a_+ + k + a_-)^{-1} S(a_+ + k + f + a_-) \\ &= S(k + a_-)^{-1} S(k) S(k + a_+)^{-1} S(k + a_+) S(k)^{-1} S(k + f + a_-) \\ &= S(k + a_-)^{-1} S(k + f) S(k)^{-1} S(k + a_-)^{-1} \\ &= S_k(a_-)^{-1} S_k(f) S_k(a_-) \end{aligned}$$

Note, that $a_-$ is independent of $f$, but not of $\mathcal{O}$. Hence the net of local algebras $(\mathfrak{A}_k(\mathcal{O}))$ is well defined for an arbitrary $\mathcal{C}^\infty$ function $k$ with an inductive limit $\mathfrak{A}_k(M)$, but there might be no global embedding of $\mathfrak{A}_k(M)$ into $\mathfrak{A}_0(M)$.

Definition of $S_k(f)$ for $k$ with supp$k$ past compact:

$$S_k(f) := S(k_1)^{-1} S(k_1 + f)$$

where $k_1 \in \mathcal{D}(M)$ with $k_1 \equiv k$ on $J_-(\mathrm{supp} f)$ (hence $a_- \equiv 0$).

Time slice axiom $\Longleftrightarrow$ $\mathfrak{A}_k(\mathcal{O}) \subset \mathfrak{A}_k(N)$

By construction: $\mathfrak{A}_k(\mathcal{O}) \subset \mathfrak{A}_0(M)$

Time slice axiom for the free field: $\mathfrak{A}_0(M) = \mathfrak{A}_0(N')$ for every neighbourhood $N'$ of a Cauchy surface.

We show: $\mathfrak{A}_0(N') \subset \mathfrak{A}_k(N)$ for $N' \subset\subset N$

Idea of proof: Compensate interaction inside $N'$:
$k' \equiv k$ on $N'$, $\mathrm{supp}k' \subset N$

Let $f \in \mathcal{D}(M)$ with $\mathrm{supp}f \subset N'$

$$S_{k,k'}(f) := S_k(-k'_1)^{-1} S_k(-k'_1 + f) = S(k_1 - k'_1)^{-1} S(k_1 - k'_1 + f)$$

$k'_1 \equiv k'$ on $J_-(\mathrm{supp}f)$, $k_1 \equiv k$ on $J_-(\mathrm{supp}f \cup \mathrm{supp}k'_1)$
$\Longrightarrow S_{k,k'}(f) = S_{k-k'}(f)$
We have: $k - k' \equiv 0$ on $N'$
$\Longrightarrow k - k' = k_+ + k_-$ with $\mathrm{supp}k_\pm \cap J_\mp(N') = \emptyset$, hence

$$S_{k-k'}(f) = S_{k_-}(f) = S(k_{-,1})^{-1} S(f) S(k_{-,1})$$

with $k_{-,1} \equiv k_-$ on $J_-(\mathrm{supp}f)$.

Consequence:

$$\alpha_{k_-} : S(f) \to S_{k_-}(f)$$

defines an injective endomorphism of $\mathfrak{A}_0(N')$ with image in $\mathfrak{A}_k(N)$.

Remaining step: surjectivity of $\alpha_{k_-}$.

Let $\tilde{N}$ be a neighbourhood of a Cauchy surface with $\tilde{N} \cap J_+(\mathrm{supp} k_-) = \emptyset$.

Define an endomorphism $\beta_{k_-}$ of $\mathfrak{A}_0(\tilde{N})$ in terms of advanced relative S-matrices $S_g^A(f) = S(f+g)S(g)^{-1}$

$$\beta_{k_-}(S(f)) = S_{k_-}^A(f)\ ,\ \mathrm{supp} f \subset \tilde{N}$$

For relatively compact $\tilde{O} \subset \tilde{N}$ we have $\beta_{k_-} = \mathrm{Ad} S(k_{-,1})$ on $\mathfrak{A}_0(\tilde{O})$, $k_{-,1} \equiv k_-$ on $J_+(\tilde{O})$.

Time slice axiom for the free theory $\Longrightarrow$

$\beta_{k_-}(\mathfrak{A}_0(\tilde{\mathcal{O}})) \subset \mathfrak{A}_0(\mathcal{O})$ with $\mathcal{O} \supset\supset J_+(\tilde{\mathcal{O}}) \cap N'$.

Choose $k_{-,1} = k_-$ on $J_+(\tilde{\mathcal{O}}) \cup J_-(\mathcal{O}) \Longrightarrow$

$$\alpha_{k_-} \circ \beta_{k_-} = \mathrm{Ad} S(k_{-,1})^{-1} S(k_{-,1}) = \mathrm{id}$$

on $\mathfrak{A}_0(\tilde{\mathcal{O}})$, $\tilde{\mathcal{O}} \subset \tilde{N}$

hence

$$\alpha_{k_-} \circ \beta_{k_-} = \mathrm{id}$$

on $\mathfrak{A}_0(\tilde{N}) = \mathfrak{A}_0(N')$ so $\alpha_{k_-}$ is surjective.

This proves the validity of the time slice axiom.

# Conclusions and Outlook

- The argument for the validity of the time slice axiom for interacting theories relies only on the functional equation of the S-matrix and the time slice axiom for the free theory. Perturbation theory is only used for the construction of the S-matrix.
- The time slice axiom allows a prezise formulation of a propagation from Cauchy surface to Cauchy surface (Schwinger equation).
- It is not obvious that the proof extends to the gauge invariant fields of a gauge theory.