AdS/CFT Correspondence and Differential Geometry

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Outline

1. Introduction: The AdS/CFT correspondence
2. Conformal Anomaly
3. AdS/CFT for field theories with $\mathcal{N} = 1$ Supersymmetry
4. Example: Sasaki-Einstein manifolds

AdS/CFT Correspondence

(Maldacena 1997, AdS: Anti de Sitter space, CFT: conformal field theory)

Witten; Gubser, Klebanov, Polyakov

- Duality Quantum Field Theory $\Leftrightarrow$ Gravity Theory
- Arises from String Theory in a particular low-energy limit
- Duality: Quantum field theory at strong coupling

  $\Leftrightarrow$ Gravity theory at weak coupling

Conformal field theory in four dimensions

$\Leftrightarrow$ Supergravity Theory on $AdS_5 \times S^5$

Anti-de Sitter space

Anti de Sitter space: Einstein space with constant negative curvature

has a boundary which is the upper half of the Einstein static universe

(locally this may be conformally mapped to four-dimensional Minkowski space)

Isometry group of AdS_5: $SO(4,2)$

AdS/CFT:

relates conformal field theory at the boundary of AdS_5

to gravity theory on $AdS_5 \times S^5$

Isometry group of S^5: $SO(6)$ $(\sim SU(4))$

AdS/CFT correspondence

- Anti-de Sitter space: Einstein space with constant negative curvature

  AdS space has a boundary

  Metric: $ds^2 = e^{2r/L}\,\eta_{\mu\nu}dx^\mu dx^\nu + dr^2$

- Isometry group of $(d+1)$-dimensional AdS space coincides with conformal group in d dimensions ($SO(d,2)$).

- AdS/CFT correspondence provides dictionary between field theory operators and supergravity fields

$$\mathcal{O}_\Delta \leftrightarrow \phi_m \;, \qquad \Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + L^2 m^2}$$

- Items in the same dictionary entry have the same quantum numbers under superconformal symmetry $SU(2,2|4)$.

Field theory side of AdS/CFT correspondence

Consider (3+1)-dimensional Minkowski space

Quantum field theory at the boundary of Anti-de Sitter space:

$\mathcal{N} = 4$ supersymmetric $SU(N)$ gauge theory ($N \to \infty$)

Fields transform in irreps of $SU(2,2|4)$, superconformal group

Bosonic subgroup: $SO(4,2) \times SU(4)_R$

- 1 vector field A_μ
- 4 complex Weyl fermions $\lambda_{\alpha A}$ ($\bar{4}$ of $SU(4)_R$)
- 6 real scalars ϕ_i (6 of $SU(4)_R$)

(All fields in adjoint representation of gauge group)

$\beta \equiv 0$, theory conformal

Supergravity side of correspondence

$(9+1)$-dimensional supergravity: equations of motion allow for
D3 brane solutions

$(3+1)$-dimensional (flat) hypersurfaces with invariance group

$\mathbb{R}^{3,1} \times SO(3,1) \times SO(6)$

Inserting corresponding ansatz into the equation of motion gives

$$ds^2 = H(y)^{-1/2} \eta_{\mu\nu} dx^\mu dx^\nu + H(y)^{1/2} dy^2$$

H harmonic with respect to y
Boundary condition: $\lim_{y\to\infty} H = 1$

$\Rightarrow$ $H(y) = 1 + \frac{L^4}{y^4}$
$L^4 = 4\pi g_s N \alpha'^2$

In addition: self-dual five-form ${F_5}^+$

Maldacena limit

For $|y| < L$: Perform coordinate transformation $u = L^2/y$

Asymptotically for u large:

$$ds^2 = L^2 \left[\frac{1}{u^2} \eta_{\mu\nu} dx^\mu dx^\nu + \frac{du^2}{u^2} + d\Omega_5{}^2 \right]$$

Metric of $AdS_5 \times S^5$

Limit:

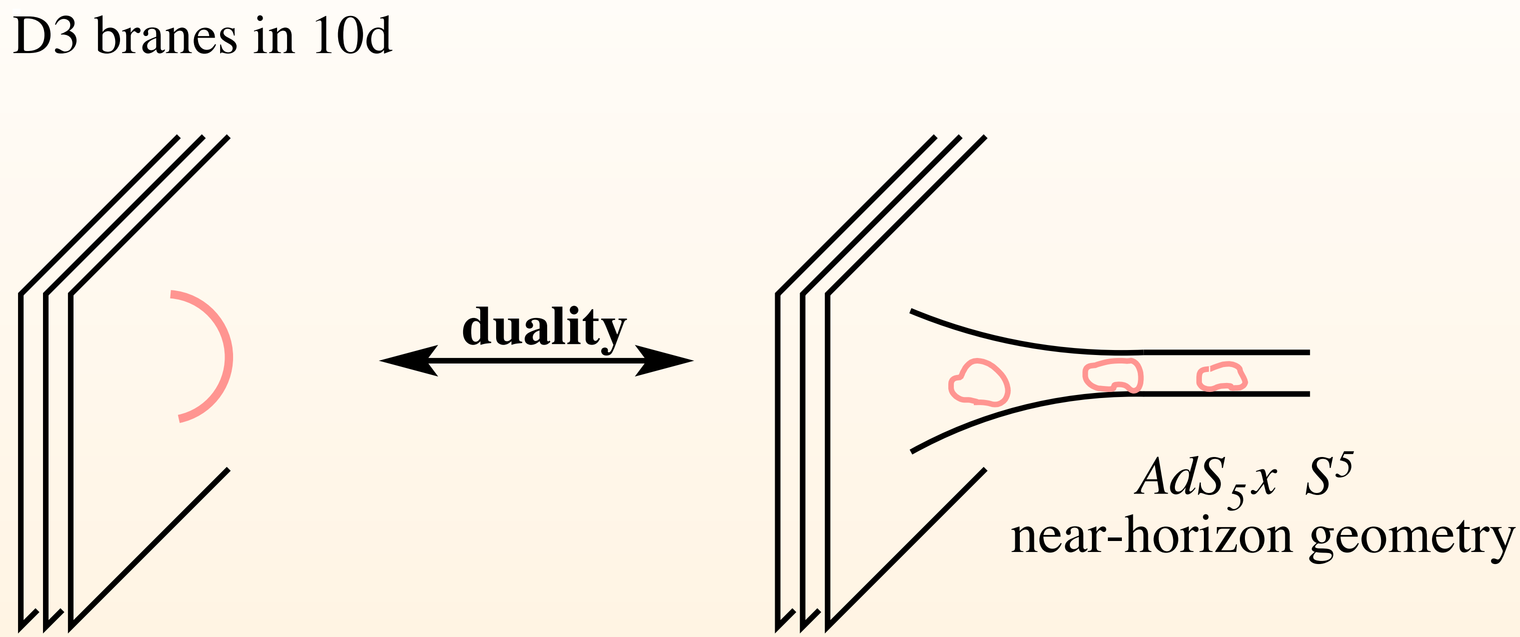
$N \to \infty$ while keeping $g_s N$ large and fixed

($l_s \to 0$)

Isometries $SO(4,2) \times SO(6)$ of $AdS_5 \times S^5$ coincide with

global symmetries of $\mathcal{N} = 4$ Super Yang-Mills theory

String theory origin of AdS/CFT correspondence



$\Downarrow$ Low-energy limit

$\mathcal{N} = 4$ SUSY $SU(N)$ gauge theory in four dimensions ($N \to \infty$)

IIB Supergravity on $AdS_5 \times S^5$

Conformal anomaly in field theory

Classical action functional $S_{Matter} = \int d^4x \sqrt{-g}\mathcal{L}_M$

Consider variation of the metric $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$

$\delta S_M = \frac{1}{2} \int d^m x \sqrt{-g}\, T^{\mu\nu} \delta g_{\mu\nu}$

$T_{\mu\nu}$ energy-momentum tensor, $T_{\mu\nu} = T_{\nu\mu}$, $\nabla^\mu T_{\mu\nu} = 0$

In conformally convariant theories: $T_\mu{}^\mu = 0$

Quantised theory: Generating functional

$$Z[g] \equiv e^{-W[g]} = \int \mathcal{D}\phi_M \, exp\left[-\int d^4x \sqrt{-g}\mathcal{L}_m\right]$$

$\delta W[g] = \int d^4x \, \langle T^{\mu\nu} \rangle \delta g_{\mu\nu}$

Consider $\delta g_{\mu\nu} = -2\sigma(x) g_{\mu\nu}$, Weyl variation: Generically $\langle T_\mu{}^\mu \rangle \neq 0$!

Conformal anomaly

In (3+1) dimensions

$$\langle T_\mu{}^\mu \rangle = \frac{c}{16\pi^2} C^{\mu\nu\sigma\rho} C_{\mu\nu\sigma\rho} - \frac{a}{16\pi^2} \frac{1}{4} \varepsilon_{\alpha\beta\gamma\delta} \varepsilon_{\mu\nu\rho\sigma} R^{\alpha\beta\mu\nu} R^{\gamma\delta\rho\sigma}$$

C Weyl tensor, $\frac{1}{4}\varepsilon_{\alpha\beta\gamma\delta}\varepsilon_{\mu\nu\rho\sigma} R^{\alpha\beta\mu\nu} R^{\gamma\delta\rho\sigma}$ Euler density

Coefficients c, a depend on $\mathcal{L}_M$

Many explicit calculation methods, for instance heat kernel

$\mathcal{N} = 4$ supersymmetric theory: $c = a = \frac{1}{4}(N^2 - 1)$

$$\langle T_\mu{}^\mu \rangle = \frac{N^2 - 1}{8\pi^2} \left(R^{\mu\nu} R_{\mu\nu} - \frac{1}{3} R^2 \right)$$

Calculation of anomaly coefficients using AdS/CFT

Henningson+Skenderis ’98, Theisen et al ’99

- Calculation of conformal anomaly using Anti-de Sitter space
- Powerful test of AdS/CFT correspondence
- Write metric of Einstein space in Fefferman-Graham form (requires equations of motion)

$$ds^2 = L^2 \left(\frac{d\rho^2}{4\rho^2} + \frac{1}{\rho} g_{\mu\nu}(x,\rho) dx^\mu dx^\nu \right)$$

$$g_{\mu\nu}(x,\rho) = \bar{g}_{\mu\nu}(x) + \rho\, g^{(2)}{}_{\mu\nu}(x) + \rho^2\, g^{(4)}{}_{\mu\nu}(x) + \rho^2 \ln \rho\, h^{(4)}{}_{\mu\nu}(x) + \ldots$$

Calculation of anomaly coefficients using AdS/CFT

- Insert Fefferman-Graham metric into five-dimensional action

$$S = -\frac{1}{16\pi G_5} \int d^5z \sqrt{|g|} \left(R + \frac{12}{L^2} \right) ,$$

$$S_\varepsilon = -\frac{1}{16\pi G_5} \int d^4x \int\limits_{\rho=\varepsilon} \frac{d\rho}{\rho} \left(a^{(0)}(x) + a^{(2)}(x)\rho + a^{(4)}(x)\rho^2 + \ldots \right)$$

- Action divergent as $\varepsilon \to 0$

- Regularisation: Minimal Subtraction of counterterm

Calculation of anomaly coefficients using AdS/CFT

- Weyl transformation gives conformal anomaly:

$$\langle T_\mu{}^\mu(x)\rangle = -\lim_{\varepsilon\to 0}\frac{1}{\sqrt{|g|}}\frac{\delta}{\delta\sigma}\left[S_\varepsilon[\bar g] - S_{ct}[\bar g]\right]$$
$$= \frac{N^2}{32\pi^2}\left(R^{\mu\nu}R_{\mu\nu} - \frac{1}{3}R^2\right)$$

- Coincides with $\mathcal{N}=4$ field theory result

- Important: Coefficient determined by volume of internal space:

  $a = \frac{\pi^3}{4}N^2 Vol(S^5) \qquad (N \gg 1)$

Field-theory coefficients a, c are related to volume of internal manifold

(S^5 for $\mathcal{N}=4$ supersymmetry)

Generalizations of AdS/CFT

Ultimate goal: To find gravity dual of the field theories in the Standard Model of elementary particle physics

First step: Consider more involved internal spaces

Example: Instead of D3 branes in flat space, consider D3 branes at the tip of a six-dimensional toric non-compact Calabi-Yau cone

Field theory: has $\mathcal{N}=1$ supersymmetry, ie. $U(1)_R$ R symmetry
(instead of the $SU(4)_R$ of $\mathcal{N}=4$ theory)

Quiver gauge theory: Product gauge group $SU(N)\times SU(M)\times SU(P)\times\ldots$

Matter fields in bifundamental representations of the gauge group

a Maximization

Conformal anomaly coefficient of these field theories can be determined by a maximization principle

In general for $\mathcal{N} = 1$ theories:

$$a = \frac{3}{32} (3 \sum_i R_i^3 - \sum_i R_i)$$

R_i charges of the different fields under $U(1)_R$ symmetry

If other $U(1)$ symmetries are present (for instance flavour symmetries), it is difficult in general to identify the correct R charges.

Result (Intriligator, Wecht 2004): The correct R charges maximise a!

Local maximum of this function determines R symmetry of theory at its superconformal point.

Critical value agrees with central charge of superconformal theory.

Supergravity side of correspondence for $\mathcal{N} = 1$ quiver theories

Metric

$$ds^2 = \frac{L^2}{r^2}\left(\eta_{\mu\nu}dx^\mu dx^\nu + dr^2 + r^2 ds^2(Y)\right)$$

with

$$ds^2(X) = dr^2 + r^2 ds^2(Y)$$

(X, ω) Kähler cone of complex dimension n ($n = 3$)

$X = \mathbb{R}^+ \times Y, \quad r > 0$

X Kähler and Ricci flat $\Leftrightarrow$ $Y = X|_{r=1}$ Sasaki-Einstein manifold

$\mathcal{L}_{r\partial/\partial r}\omega = 2\omega \quad \rightarrow \quad \omega$ exact: $\omega = -\frac{1}{2}d(r^2\eta)$, $\quad \eta$ global one-form on Y

Supergravity side of correspondence for $\mathcal{N} = 1$ quiver theories

- Kähler cone X has a covariantly constant complex structure tensor $\mathcal{I}$
- Reeb vector $K \equiv \mathcal{I}(r\frac{\partial}{\partial r})$
- Constant norm Killing vector field
- Reeb vector dual to $r^2\eta \to \eta = \mathcal{I}(\frac{dr}{r})$
- Reeb vector generates the AdS/CFT dual of $U(1)_R$ symmetry
- Sasaki-Einstein manifold $U(1)$ bundle over Kähler-Einstein manifold, $U(1)$ generated by Reeb vector

Geometrical equivalent of a maximization

Martelli, Sparks, Yau 2006

Variational problem on space of toric Sasakian metrics

toric cone X

real torus T^n acts on X preserving the Kähler form –

supersymmetric three cycles

Einstein-Hilbert action on toric Sasaki Y reduces to volume function $vol(Y)$

Kähler form: $\omega = \sum_{i=1}^{3} dy_i \wedge d\phi_i$

Symplectic coordinates $(y_i \phi_i)$,
ϕ_i angular coordinates along the orbit of the torus action

For general toric Sasaki manifold define vector $K' = \sum_{i=1}^{3} b_i \frac{\partial}{\partial \phi_i}$

$\Rightarrow vol[Y] = vol[Y](b_i)$

Geometrical equivalent of a maximization

Reeb vector selecting Sasaki-Einstein manifold corresponds to those b_i which minimise volume of Y

Volume minimization $\Rightarrow$ Gravity dual of a maximization

Volume calculable even for Sasaki-Einstein manifolds for which metric is not known (toric data)

Example: Conifold

Base of cone: $Y = T^{(1,1)}$, $\quad T^{(1,1)} = (SU(2) \times SU(2))/U(1)$

Symmetry $SU(2) \times SU(2) \times U(1)$, topology $S^2 \times S^3$

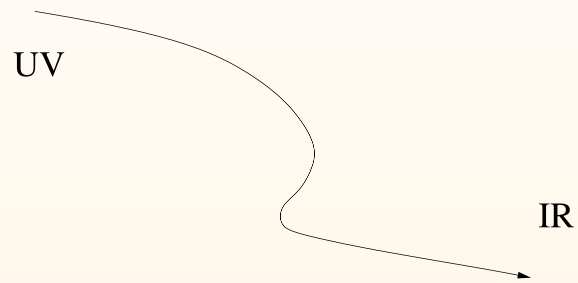
Dual field theory has gauge group $SU(N) \times SU(N)$

$$Vol(T^{(1,1)}) = \frac{16}{27}\pi^3$$

- can be calculated using volume minimisation as described
- gives correct result for anomaly coefficient in dual field theory

There exists an infinite family of Sasaki-Einstein metrics $Y^{p,q}$

C-Theorem (Zamolodchikov 1986) in $d = 2$: $\dot{C}(g^i) \leq 0$, $C = C(g^i(\mu))$



So far no field-theory proof in $d = 4$ exists

There is a version of the C theorem in non-conformal generalisations of AdS/CFT

Metric: $ds^2 = e^{2A(r)}\, \eta_{\mu\nu} dx^\mu dx^\nu + dr^2$

C-Function:

$$C(r) = \frac{c}{A'(r)^3}$$

Outlook

To investigate non-conformal examples of gauge theory/gravity duality

with methods of differential geometry

Conclusions

- AdS/CFT provides a powerful relation between gauge theory and gravity.
- It originates from string theory.
- Calculation of conformal anomaly provides powerful check.
- Generalisations to less symmetric field theories are possible.
- Further generalisations will provide
  - new insights into the structure of string theory
  - new non-perturbative tools to describe field theories.