Special geometry with solvable Lie groups

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Special geometry
Lie groups' actions
Six dimensions
Seven dimensions
Nilpotent/Solvable Lie groups
Nilmanifolds
Prototypical example
Solvmanifolds
Non-compact homogeneous Einstein spaces
Half-flatness
Geometry with torsion
Spinors
Strings attached
'Simultaneous' structures
End

Under the auspices of a famous scientist from Hamburg

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- Lie groups' actions
- Six dimensions
- Seven dimensions

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1. **Special geometry**
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   - Prototypical example
   - Solvmanifolds
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3. **Geometry with torsion**
   - Spinors
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4. **End**

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Holonomy groups of Ricci-flat metrics

dim	6	7	8
group	$SU(3)$	G_2	$Spin(7)$

local examples: easy(-ish) to find

complete/compact examples: harder, but fortunately the explicit knowledge of the metric is often unnecessary

In dims 6, 7, 8 interesting structures are determined by differential forms lying in open orbits under the action of $GL(n, \mathbb{R})$

For instance, in the intermediate dimension a certain 3-form determines the *whole* geometry

Maximal subgroups of G_2: $SO(3)$, $SO(4)$, $SU(3)$

And $G_2 \subset SO(8) \rightsquigarrow Spin(7)$-, $PSU(3)$-, $Sp(2)Sp(1)$-geometry.

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G-structures

$Spin(6) = SU(4)$ acts transitively on S^7

$$\frac{SO(6)}{SU(3)} = \frac{SO(7)}{G_2} = \frac{SO(8)}{Spin(7)} = \mathbb{RP}^7$$

Different sets of reductions are parametrised by the same space, which by the way admits G_2 structures

Related to this

- $S^6 = G_2/SU(3)$
- $(S^6, g_{\mathrm{round}}) \subset \mathbb{R}^7$ has an almost complex structure J inherited from the *vector cross product* on $\mathbb{R}^7$
- J is nearly Kähler

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Examples of interaction

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- Hypersurface theory $X^n \hookrightarrow Y^{n+1}$, quotients X/S^1, and the like

- conical singularities constructed from NK structures: the cone of $SU(2)^3/SU(2)$ deforms to a complete smooth holonomy metric on $Y \cong \mathbb{R}^4 \times S^3$ [Bryant-Salamon]

  Similarly for $Ber = SO(5)/SO(3), \quad AW = SU(3)/U(1)$

- (M^6, g) NK $\Longrightarrow$ the sine cone $dt^2 + (\sin^2 t)g$ has weak holonomy G_2 (so Einstein). Its singularities at $t = 0, \pi$ approximate G_2-holonomy cones [Acharya & al], see [Fernández & al] too

  This example has the flavour of Killing spinors

- ALC singularities of [Gibbons–Lü–Pope–Stelle]

Tensors and representations

Let (X^d, g) be Riemannian and ϕ a tensor, define

$$G = \{a \in SO(d) : a^*\phi = \phi\}$$

so $\Lambda^2 T^* X = \mathfrak{so}(d) = \mathfrak{g} \oplus \mathfrak{g}^\perp$ and $Hol(g) \subseteq G \iff \nabla\phi = 0$

By analogy with the complex case, these are often referred to as *integrable G-geometries*

- $\nabla\phi$ is identified with the intrinsic torsion, an element in

$$T^* \otimes \mathfrak{g}^\perp \cong \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \ldots \oplus \mathcal{W}_N$$

with N irreducible components. Notice $\frac{\mathfrak{so}(d)}{\mathfrak{g}} = \mathbb{R}^7$ when $d = 6, 7, 8$

d	ϕ	G	N
$2m$	almost complex structure J	U(m)	4
$2m$	non-degenerate 2-form σ	U(m)	4
7	positive generic 3-form	G_2	4
8	positive generic 4-form	$Spin(7)$	2
$4k$	quaternionic 4-form	Sp(k)$Sp(1)$	6

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Six dimensions

- g Riemannian metric
- J orthogonal almost complex structure
$$J \in End\,TM : J^2 = -1,\ g(JX, JY) = g(X, Y)$$
- σ non-degenerate 2-form
$$\sigma(X, Y) = g(JX, Y)$$
- $\Psi \in \Lambda^{3,0} T^* M$ a complex volume form
$$\sigma \wedge \Psi = 0, \qquad \Psi \wedge \bar{\Psi} = \tfrac{4}{3} i \sigma^3$$
- $\psi^+ = \mathrm{Re}\,\Psi$ with open orbit in $\Lambda^3 \mathbb{R}^6$
(determines J, hence $\psi^- = J\psi^+ = \mathrm{Im}\,\Psi$)

$\Longrightarrow$ *Complex and symplectic aspects are linked:*
the structure is determined by choosing ψ^+, σ only, for
$$SL(3, \mathbb{C}) \cap Sp(6, \mathbb{R}) = SU(3)$$

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$SU(3)$-intrinsic torsion

The holonomy group $Hol(g)$ is contained in $SU(3)$ iff all forms are constant for the Levi–Civita connection

$$\nabla\sigma = 0, \qquad \nabla\psi^\pm = 0$$

Obstruction:

$$\nabla J \in T^* \otimes \mathfrak{su}(3)^\perp \cong \mathcal{W}_1^\pm \oplus \mathcal{W}_2^\pm \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 \oplus \mathcal{W}_5$$

where $\mathcal{W}_j$ are the so-called ‘Gray–Hervella classes’

The intrinsic torsion is completely determined by the exterior derivatives of σ, ψ^+ and ψ^- ($n > 3$ only σ, ψ^+!)

$$\nabla J = 0 \iff \text{all forms are closed: } d\sigma = 0,\ d\psi^\pm = 0$$

$\rightsquigarrow$ M is a Calabi–Yau manifold

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almost Hermitian taxonomy

comp	$\dim_{\mathbb{R}}$	$U(3)$-module	$SU(3)$-module	
$\mathcal{W}_1^{\pm}$	1+1	$[\![\Lambda^{3,0}]\!]$	$\mathbb{R}$	$\mathbb{R}$
$\mathcal{W}_2^{\pm}$	$8+8$	$[\![V]\!]$	$\mathfrak{su}(3)$	$\mathfrak{su}(3)$
$\mathcal{W}_3$	12	$[\![\Lambda_0^{2,1}]\!]$	$\mathcal{S}^{2,0}$	
$\mathcal{W}_4$	6	Λ^1	Λ^1	
$\mathcal{W}_5$	6	Λ^1	Λ^1	

For instance

- $\nabla J \in \mathcal{W}_3 \oplus \mathcal{W}_4 \iff N_J = 0$ e.g. $\mathbb{C}^3,\ G \times T^m$
- $\nabla J \in \mathcal{W}_1 \iff M$ is nearly Kähler $\mathcal{Z}(S^4)$
- $\nabla J \in \mathcal{W}_2 \iff d\sigma = 0$ $KT = S^1 \times H_3/\Gamma$
- $\nabla J \in \mathcal{W}_4 \iff$ loc. conformally Kähler $SU(2) \times U(1)$

You name it ...

G_2 structures

On a 7-manifold Y with tangent spaces $T_y Y = \mathbb{R}^6 \oplus \mathbb{R}$ and $SU(3) \times \{1\}$ structure, define

$$\varphi = \sigma \wedge e^7 + \psi^+$$

$$*\varphi = \psi^- \wedge e^7 + \tfrac{1}{2}\sigma^2$$

In terms of an ON basis

$$\varphi = e^{127} + e^{347} + e^{135} + e^{425} + e^{146} + e^{236} + e^{567}$$

[Engel, Reichel] Stab$(\varphi) = G_2$

$\implies$ open $GL(7, \mathbb{R})$-orbit in $\Lambda^3 T^* Y$

[Bryant] Such a φ determines the metric g and $*\varphi$

[Fernández–Gray] $Hol(g) \subseteq G_2 \iff d\varphi = 0, d{*\varphi} = 0$

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The intrinsic torsion of a G_2 structure

$$\nabla\varphi \in \Lambda^1 \otimes \mathfrak{g}_2^\perp = \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4$$

is encoded into the exterior derivatives $d\varphi, d{*}\varphi$

class	type	conditions
—	G_2 holonomy	$d\varphi = 0 = d{*}\varphi$
$\mathcal{X}_1$	weak holonomy	$d\varphi = \lambda {*}\varphi$
$\mathcal{X}_4$	conformally G_2	$\begin{cases} d{*}\varphi = 4\theta \wedge {*}\varphi \\ d\varphi = 3\theta \wedge \varphi \end{cases}$
$\mathcal{X}_2$	calibrated	$d\varphi = 0$
$\mathcal{X}_1 \oplus \mathcal{X}_3$	cocalibrated	$d{*}\varphi = 0$
$\mathcal{X}_1 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4$	G_2T	$d{*}\varphi = \vartheta \wedge {*}\varphi$

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Nilpotency

G is *k-step nilpotent* iff $\exists k : \{0\} \neq \mathfrak{g}^{k-1} \supset \mathfrak{g}^k = \{0\}$ where

$$\mathfrak{g}^0 = \mathfrak{g}, \quad \mathfrak{g}^i = [\mathfrak{g}^{i-1}, \mathfrak{g}] \qquad \text{(lower central series)}$$

e.g. 1-step = Abelian, 2-step $\iff$ $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{z}$

- Classification: finitely many isomorphism types for $\dim_{\mathbb{R}} \leqslant 6$, continuous families in $\dim_{\mathbb{R}} = 7$. Afterwards ?
- G has rational structure constants $\Longrightarrow \exists \Gamma \ : \ M = G/\Gamma$ is compact [Malcev]

The *compact* quotient $M = G/\Gamma$ of a real 1-connected nilpotent Lie group G by a lattice Γ is called a nilmanifold

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Features

Let $M = G/\Gamma$ be a nilmanifold

[Nomizu] $H^k_{dR}(M) \cong H^k(\mathfrak{g})$

where the latter is the cohomology of the Chevalley-Eilenberg complex $(\bigwedge \mathfrak{g}^*, d)$ of G-invariant forms

By the way, what about $H^{*,*}_{\bar\partial}(M) \overset{?}{\cong} H^{*,*}_{\bar\partial}(\mathfrak{g}^{\mathbb{C}})$ $\rightsquigarrow$ [Console-Fino, et al.]

[Sullivan] $\bigwedge \mathfrak{g}^*$ is a minimal model of M

[Hasegawa] M is formal $\iff$ G is Abelian and M is a torus

'formal' roughly means $\bigwedge \mathfrak{g}^*$ captures the homotopy type of M

examples: compact Kähler mfds, homog. spaces of max. rank,
compact simply conn. mfds of dim $\leqslant 6$

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Left-invariant structures

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A nilpotent Lie group N^n may or not admit left-invariant complex or symplectic structures (in contrast to compact simple)

[Benson-Gordon, ...] Besides tori, nilmanifolds N/Γ never admit Kähler metrics

N real 1-connected nilpotent Lie group
$\iff \exists$ a basis $\{e^1, \ldots, e^n\}$ of left-invariant 1-forms such that

$$de^i \in \Lambda^2\langle e^1, \ldots, e^{i-1}\rangle, \quad i = 1, \ldots, n$$

For fixed metric on any N^6, almost Hermitian structures define points of $\dfrac{SO(6)}{U(3)} = \mathbb{CP}^3$, described by [Abbena & al]

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Example (the Iwasawa manifold)

The complex Heisenberg group

$$G = \left\{ \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} : z_i \in \mathbb{C} \right\} = H_3$$

defines a nilmanifold $M = G/\Gamma$ where Γ is the subgroup with $z_\alpha \in \mathbb{Z}[i]$.
Mapping to (z_1, z_2) realises M as a T^2-bundle over T^4 (similar to twistor fibration over X^4)

The real basis (e^i) of $T_e^* G \cong \mathfrak{g}^*$ with

$$dz_1 = e^1 + ie^2,\ dz_2 = e^3 + ie^4,\ -dz_3 + z_1 dz_2 = e^5 + ie^6 \ \in \Lambda^{1,0}$$

satisfies

$$de^i = \begin{cases} 0, & 1 \leqslant i \leqslant 4 \\ e^{13} + e^{42}, & i = 5 \\ e^{14} + e^{23}, & i = 6 \end{cases}$$

written $\mathfrak{g} = (0, 0, 0, 0, e^{13} + e^{42}, e^{14} + e^{23})$

The Kähler form $\sigma = -e^{12} - e^{34} + e^{56}$ defines an $SU(3)$ structure on $Iwa = H_3/\Gamma$ with $d\sigma = \psi^+$

First explicit solutions of the Hitchin flow (via nilmanifolds!):

Proposition (myself)

A fibre product $Iwa \times_t \mathbb{R}_+$ admits a metric with holonomy G_2 induced from

$$\varphi = \sigma(t) \wedge dt + \psi^+(t)$$

by deforming the standard half-flat $SU(3)$ structure (Iwa, σ_0, ψ_0) as follows:

$$\begin{aligned} \psi^+(t) &= \psi_0^+ + x(t) d(e^{56}) \\ \tfrac{1}{2}\sigma(t)^2 &= \tfrac{1}{2}\sigma_0^2 + y(t) e^{1234} \end{aligned} \quad \text{with} \quad \begin{cases} \dot{x}(t) = \frac{1}{\sqrt{y+1}} \\ \dot{y}(t) = -4x \end{cases}$$

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Solvability

G is solvable $\iff \exists k : \{0\} \neq \mathfrak{g}_{k-1} \supset \mathfrak{g}_k = \{0\}$ where

$$\mathfrak{g}_0 = \mathfrak{g}, \quad \mathfrak{g}_i = [\mathfrak{g}_{i-1}, \mathfrak{g}_{i-1}] \quad \text{(derived series)}$$

The quotient $M = G/\Gamma$ of a real 1-connected solvable Lie group G by a discrete co-compact subgroup Γ, or

G with a left-invariant metric is called a solvmanifold

- $(G, g_{\text{invariant}})$ 1-connected, flat $\Longrightarrow$ solvable [Milnor]
- symplectic, unimodular $\Longrightarrow$ solvable [Chu]

$M = G/K$ symm. space of non-compact type $\Longrightarrow G = KAN$ Iwasawa decomposition, M isometric to $S = AN$

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Features

$M = G/\Gamma$ compact solvmanifold, G simply connected and completely solvable (= *ad* has real eigenvalues)

[Hattori] $\bigwedge \mathfrak{g}^*$ is quasi-isomorphic to $\Omega_{dR}(G/\Gamma)$, hence a model of M

[Benson-Gordon] G completely solvable, G/Γ compact Kählerian solvmanifold $\iff$ M diffeo to a torus

[Hasegawa] (cf. [Cortés-Baues])
compact solvmfd is Kählerian $\iff$ finite quotient of complex torus, and a complex torus bundle over a complex torus

$$\mathbb{C}^l/\mathbb{Z}^{2l} \longrightarrow M = \mathbb{T}^{l+k}/\Delta \longrightarrow \mathbb{C}^k/\mathbb{Z}^{2k} \text{ holomorphic fibration}$$

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Solvable examples

[Gibbons & al] Incomplete Ricci-flat metrics with $Hol \subseteq G_2$ and 2-step nilpotent isometry groups N^6 acting on orbits of codim 1

POINT IS

Theorem (Fino-myself)

these are (loc.) conformally isometric to homogeneous metrics on solvable Lie groups

$$S = \widetilde{\Gamma \backslash N} \times \mathbb{R}$$

built from N

Proposition (ditto)

Classification of nilpotent (N^6, σ, ψ^+) whose rank-one solvable extension has $\varphi = \sigma \wedge e^7 + \psi^+$ conformally G_2

Actually (S, φ) is conformally $G_2 \iff N$ either T^6 or 2-step nilpotent (but $\neq H_3 + H_3$)

Can think of $\Gamma \backslash N$ as a torus bundle over a torus [Palais-Stewart]

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Non-compact homogeneous Einstein spaces

- A solvmanifold $(S, g_{\text{invariant}})$ is a homogeneous Einstein space with non-positive scalar curvature

- All known examples of non-compact, non-flat, homogeneous Einstein spaces G/K have K maximal compact, i.e. are isometric to a $(S, g_{\text{invariant}})$ (*conjecture of Alekseevskii*)

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Examples I

• S unimodular, solvable $\Longrightarrow$ every left-inv. Einstein metric is flat [Dotti]

• G unimodular with inv. Kähler structure $\Longrightarrow$ flat, $G = A \ltimes [G, G]$, both factors Abelian [Hano]

• Homog. Einstein, Ricci-flat $\Longrightarrow$ flat [Alekseevskii-Kimelfeld]

• K-E solvmanifolds are biholomorphic to bounded symmetric domains with Bergmann metric [D'Atri-Dotti]

• Classification of QK solvmanifolds [Alekseevskii-Cortés], via [Lauret]

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Examples II: Einstein solvable extensions of NLAs

- (standard) Einstein solvmanifolds are – up to isometry – *metric solvable extensions of Iwasawa type*

$$\mathfrak{s} = [\mathfrak{s}, \mathfrak{s}] \oplus \mathfrak{a} = \mathfrak{n} \oplus \mathfrak{a}$$

$$ad_{\mathfrak{a}} : \mathfrak{n} \to \mathfrak{n} \text{ self-adjoint and pairwise commuting}$$

$$\exists A \in \mathfrak{a} : ad_A \text{ positive-definite}$$

- Can reduce to $\mathfrak{a} = \mathbb{R}H$ (extension of rank 1), with $\langle H, \mathfrak{n} \rangle = 0$, $\|H\| = 1$ and $[X, Y] = [X, Y]_{\mathfrak{n}}$, $[H, X] = DX$ for some $D \in \mathrm{Der}(\mathfrak{n})$ [Heber], [Heintze]

- Einstein solvmanifolds are standard [Lauret]

If $\dim \mathfrak{n} \leqslant 6$ there is always a rank-one Einstein solvable extension [Lauret, Will]

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Concretely, please

Take $\mathfrak{s} = (0, 0, \frac{2}{5} m\, e^{15}, \frac{2}{5} m\, e^{25}, 0, \frac{2}{5} m\, e^{12}) \oplus \mathbb{R} e^7$ with

$$\begin{cases} de^1 = -\frac{3}{5} me^{17} & de^2 = -\frac{3}{5} me^{27}, \\ de^3 = \frac{2}{5} me^{15} - \frac{6}{5} me^{37}, & de^4 = \frac{2}{5} me^{25} - \frac{6}{5} me^{47}, \\ de^5 = -\frac{3}{5} me^{57}, & de^6 = \frac{2}{5} me^{12} - \frac{6}{5} me^{67}, \quad de^7 = 0 \end{cases}$$

Besides an Einstein metric $\sum (e^i)^2$ (with Ric < 0),

Proposition (Fino-myself)

There is a G_2-holonomy structure on $S \cong \mathbb{R} \times \mathcal{T}$, where

$$\begin{array}{ccc} T^3 & \longrightarrow & \mathcal{T} \\ & & \downarrow \\ & & T^3 \end{array}$$

the base is span$\{e_1, e_2, e_5\}$, the fibre span$\{e_3, e_4, e_6\}$

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Proposition (ditto)

$\mathcal{T} = N/\Gamma$ equipped with $SU(3)$ forms

$$\sigma_0 = e^{56} - e^{23} + e^{14}, \quad \psi_0^+ = -e^{345} + e^{136} + e^{246} + e^{125}$$

flows to the Ricci-flat metric on $\mathcal{T} \times \mathbb{R}$

$$g = (1 - mt)^{4/5} g_{\text{fibre}} + (1 - mt)^{-2/5} g_{\text{base}} + dt^2,$$

in terms of the flat metrics on fibre- and base tori

Oh, and: this and the previous metric are essentially the same, albeit arising rather differently (ie via *Einstein solvable extensions*, and using the *evolution equations* described below)

Half-flatness

An $SU(3)$ structure (ψ^+, σ) is called half-flat if

$$d\psi^+ = 0 \quad \text{and} \quad d(\sigma \wedge \sigma) = 0$$

- 21/42 of the torsion vanish
- $\mathcal{W}_1^+, \mathcal{W}_2^+, \mathcal{W}_4, \mathcal{W}_5$ are zero
- akin to 'ASD + Ric $= 0$' in dim 4 (but much weaker)

Theorem (Swann-myself)

Classification of invariant half-flat $SU(3)$ structures on nilpotent Lie groups N^6 such that $N \times S^1$ is G_2T

Why on earth the need for another $SU(3)$-class?

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The quest for G_2-holonomy metrics

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Assume M^6 is compact with $SU(3)$ structure $\sigma(t), \psi^+(t)$ depending on t

Let $Y^7 = M \times_t (a, b)$ bear $\varphi = \sigma(t) \wedge dt + \psi^+(t)$

$$0 = d\varphi = \left(d\sigma - \frac{\partial \psi^+}{\partial t} \right) \wedge dt + d\psi^+$$

$$0 = d{*}\varphi = \left(d\psi^- + \sigma \wedge \frac{\partial \sigma}{\partial t} \right) \wedge dt + \tfrac{1}{2} d(\sigma \wedge \sigma)$$

A half-flat M^6 evolves to a structure on Y^7 with $Hol \subseteq G_2$
Hamiltonian theory guarantees solution [Hitchin]

Special case: $d\sigma = a\psi^+$ and $d\psi^- = b\sigma^2$ (like S^6)

Solving these PDEs is hard, but… see p.17

G_2-holonomy v spinors

G_2 is the isotropy in *Spin*(7) of a spinor $\eta \in \Delta_7 \cong \mathbb{R}^8$

The G_2-fundamental form $\varphi \in \Lambda^3\mathbb{R}^7$ is defined as

$$\varphi(X, Y, Z) = \langle X \cdot Y \cdot Z \cdot \eta, \eta \rangle$$

Remember

- (M^7, φ) has holonomy $G_2 \iff \exists\, \eta_0 \in \Delta_7 : \nabla \eta_0 = 0$
- (M^7, φ) is conformally $G_2 \iff Hol(e^{2f} g) \subseteq G_2$, for some f

Fact:

the number of parallel spinors determines the amount of symmetry of the manifold [Wang]

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More symmetry: skew torsion

Q: Given (M^7, φ) with $Hol(g) = G_2$, are there other parallel spinors

$$\widetilde{\nabla}\eta = 0$$

besides η_0 ?

A: Yes (sometimes many), if M^7 is a solvmanifold

To find more we are forced to look for different $\widetilde{\nabla}$ as well, say metric connections with skew-symmetric torsion ([Cartan], revamped by [Ivanov-Friedrich])

$$\nabla^T = \nabla + Torsion = \nabla + \tfrac{1}{2}T, \qquad T \in \Lambda^3\mathbb{R}^7$$

Precisely:

$T(X, Y, Z) = g(\nabla^T_X Y - \nabla^T_Y X - [X, Y], Z)$ is skew in X, Z, Y

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Strings, anyone?

This is meant to hint at the type-II string equations with *constant dilaton and no fluxes*,

$$\mathrm{Ric}^{\nabla^T} = 0 \qquad d{*}T = 0$$

$$\nabla^T \eta = 0 \qquad T \cdot \eta = 0$$

[Strominger] A Riemannian manifold (X^d, g, T, η, f) with

$$T \quad \text{3-form}, \quad \eta \quad \text{spinor field}, \quad f \quad \text{function},$$

$$\nabla^T = \nabla + \tfrac{1}{2} T \quad \text{metric connection with } \textit{skew} \text{ torsion } T,$$

yield (partial) solutions to the equations

[Agricola & al] A full solution forces $T = 0, \nabla^T = \nabla$ and scal$= 0$

But (there's a but)...

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The good news

Theorem (Agricola-Fino-myself)

The equation $\nabla^T \eta = 0$ has the following solutions on the previous solvmanifold

$$S \cong \mathbb{R} \times \mathcal{T}, \quad \mathcal{T} = T^3\text{-bundle over } T^3 :$$

- *a family of parallel spinors*

$$\eta_{r,s} = (0,0,0,0,r,s,-r,s), \qquad r/s \in \mathbb{R} \cup \{\infty\}$$

  and a family of torsion connections $\nabla + \frac{1}{2} T_{r,s}$,

$$T_{r,s} = const \left[\lambda_{r,s}(\psi^+ - 6e^{125}) + \mu_{r,s}(\psi^- + 3e^{346}) \right],$$

  deforming the Levi-Civita.

$$\left(\lambda = \frac{r^2 - s^2}{2(r^2+s^2)}, \quad \mu = \frac{(r-s)^2}{r^2+s^2} \quad \text{homogeneous} \right)$$

- *six 'isolated' solutions $(\nabla^{T_\alpha}, \eta_\alpha) : \nabla^{T_\alpha} \eta_\alpha = 0$*

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On the other solvmanifolds of [Fino-myself] admit either no additional parallel spinors (rigidity) or complex solutions.

Quick proof:

• Let $\mathbb{V}$ be the subspace of $\Lambda^3\mathbb{R}^7$ spanned by the simple forms appearing in $\psi^{\pm}, \sigma \wedge e^7$, hence $\dim \mathbb{V} = 11 < 35$

• Take $H \in \mathbb{V}$, lift $\nabla^H = \nabla + \frac{1}{2} H$ to the spin bundle, so that parallel spinors are solutions to

$$\nabla^H_X \eta = \nabla_X \eta + (X \lrcorner H) \cdot \eta = 0, \qquad \forall X$$

• The endomorphism $(e_i \lrcorner H) \cdot$ has block structure $\left(\begin{smallmatrix} 0 & * \\ * & 0 \end{smallmatrix}\right)$

• For $i = 7$: $\quad Ker\,(\nabla_{e_7} + e_7 \lrcorner H) = Ker\,(e_7 \lrcorner H)$

(to be completely honest, ∇^H is a'conformal' Levi-Civita)

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A moduli space of sorts

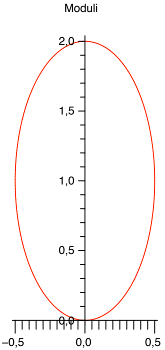
$\lambda = \lambda_{r,s},\ \mu = \mu_{r,s}, \quad r/s \in \mathbb{RP}^1$

Each point on the conic $(\mu - 1)^2 + 4\lambda^2 = 1$ corresponds 1-1 to

- a torsion connection $\nabla^{T_{r,s}}$ plus a parallel spinor $\eta_{r,s}$
- a choice of $\psi^+ + i\psi^- \in \Lambda^{3,0} T^* N^6$
- a G_2 structure $\varphi_{r,s} =$

$$rs\,\psi^+ + \frac{r^2-s^2}{2}\psi^- + \frac{r^2+s^2}{2}\sigma \wedge e^7$$

of expected type $\mathcal{X}_{1+3+4}$, generically



NB: the metric is the same, i.e.
all $\varphi_{r,s}$ induce only one Riemannian structure!

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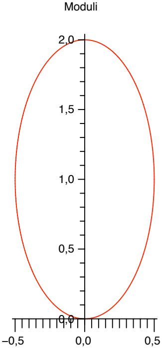
A moduli space of sorts

G_2-analysis: the 3-form $\varphi_{r,s}$

- has $Hol = G_2$ when $r = s$
  $\eta_{r,r} \ \sim \ \eta_0$ ($\rightsquigarrow$ origin)
- has type $\mathcal{X}_{3+4}$ for $r = -s$
  ($\rightsquigarrow$ top point)
- has $\mathcal{X}_3 \neq 0$ always
  (bar $\varphi_{r,r}$, clearly)



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What(ever) next?

- Relax the extension hypotheses (= how to build S from N)

  e.g. forget Einstein

- Pick nilpotent Lie groups N^6 with step-length $\geqslant 3$

  i.e. more bundled structures

- Let T roam the full space $\Lambda^3\mathbb{R}^7 \rightsquigarrow$ expect more examples

- Consider different G_2-types on S

Upshot: nil- and solvmanifolds are quite interesting

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that's really it, thanks

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