Calabi-Yau Metrics and the Spectrum of the Laplace Operator

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1. Introduction
   - String Theory
   - Interesting Things to Calculate
2. Calabi-Yau Metrics
3. The $\mathbb{Z}_5 \times \mathbb{Z}_5$ Quotient
4. The Laplace Operator
5. Pretty Pictures

String Theory

- Field theory (Supergravity) limit of string theory:

$$M_{\mathrm{Pl}} > M_{\mathrm{GUT}}$$

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- Laplace equation on the threefold

$$\Delta\, \Phi_i^{(6)} = \lambda_i \Phi_i^{(6)}$$

determines KK modes.

Wish List

- Zero modes $\lambda_n = 0$ determine the light 4-d particles.
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- Massive modes $\lambda_n > 0$.
- Higher-order couplings.
  A^∞ products.

1. **Introduction**
2. **Calabi-Yau Metrics**
   - Kähler Geometry
   - Donaldson's Algorithm
   - Implementation
   - Testing the Metric
3. **The $\mathbb{Z}_5 \times \mathbb{Z}_5$ Quotient**
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Kähler Metrics on the Quintic

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$$Q_F = \left\{ z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 = 0 \right\} \subset \mathbb{P}^4$$

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- The metric is completely determined by the Kähler potential $K(z, \bar{z})$:

$$g_{i\bar{j}}(z, \bar{z}) = \partial_i \bar{\partial}_{\bar{j}} K(z, \bar{z})$$
$$\omega = g_{i\bar{j}}(z, \bar{z})\, \mathrm{d}z^i\, \mathrm{d}\bar{z}^{\bar{j}} = \partial\bar{\partial} K(z, \bar{z}).$$

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- Locally, K is a real function.
- ω is a $(1, 1)$-form.

Fubini-Study Metric

Unique $SU(5)$ invariant Kähler metric on $\mathbb{P}^4$

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$$K_{\mathrm{FS}} = \ln \sum_{\alpha,\bar{\beta}=0}^{4} h^{\alpha\bar{\beta}} z_\alpha \bar{z}_{\bar{\beta}}$$

with h a hermitian 5×5 matrix.

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Restrict to $Q \subset \mathbb{P}^4$. Not Ricci flat.

Donaldson's Ansatz

Let's try [Donaldson]

$$K(z,\bar{z}) = \ln \sum_{\substack{\sum i_\ell = k \\ \sum \bar{j}_\ell = k}} h^{(i_1,\ldots,i_k),(\bar{j}_1,\ldots,\bar{j}_k)} \underbrace{z_1^{i_1}\cdots z_k^{i_k}}_{\text{degree } k} \underbrace{\bar{z}_1^{\bar{j}_1}\cdots \bar{z}_k^{\bar{j}_k}}_{\text{degree } k}$$

for some hermitian $N \times N$ matrix h

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$$N = \binom{5+k-1}{k} = \{\#\,\text{deg}\,k\ \text{monomials}\}$$

Technicalities

On the quintic $z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 = 0$. So not all monomials are independent in degrees $k \geq 5$.

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Let s_α be a basis for

$$\mathbb{C}[z_0, \dots, z_4] \Big/ \left\langle z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 = 0 \right\rangle \Big|_{\text{degree } k}$$

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More Technical

- s_α are sections of $\mathcal{O}_Q(k)$

$$0 \to H^0\big(\mathbb{P}^4, \mathcal{O}(k-5)\big) \longrightarrow H^0\big(\mathbb{P}^4, \mathcal{O}(k)\big) \longrightarrow H^0\big(Q, \mathcal{O}_Q(k)\big) \to 0$$

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- Metric on the line bundle $\mathcal{O}_Q(k)$

$$(\sigma, \tau) = \frac{\sigma(z)\bar{\tau}(\bar{z})}{\sum h^{\alpha\bar{\beta}} s_\alpha(z)\bar{s}_{\bar{\beta}}(\bar{z})}$$

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Balanced Metrics

h is "balanced" if the matrices representing the metrics coincide, that is:

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Theorem

Let h be the balanced metric for each k. Then the sequence of metrics

$$\omega_k = \partial\bar\partial \ln \sum h^{\alpha\bar\beta} s_\alpha \bar{s}_{\bar\beta}$$

converges to the Calabi-Yau metric as $k \to \infty$.

T-Operator

How to solve

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Donaldson's T-operator:

$$\begin{aligned} T(h)_{\alpha\bar\beta} &= \langle s_\alpha, s_\beta\rangle \\ &= \int_Q \frac{s_\alpha \bar{s}_{\bar\beta}}{\sum h^{\alpha\bar\beta} s_\alpha(z)\bar{s}_{\bar\beta}(\bar z)}\, \mathrm{dVol} \end{aligned}$$

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One can show that iterating $T(h_n)^{-1} = h_{n+1}$ converges!
Fixed point is balanced metric.

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- The approximate Calabi-Yau metric is

$$g_{i\bar{j}} = \partial_i \bar{\partial}_{\bar{j}} \ln \sum s_\alpha h^{\alpha\bar{\beta}} \bar{s}_{\bar{\beta}}$$

Details

- *Exact* Calabi-Yau volume form

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- Integrate by summing over random points. [Douglas,Karp,Lukic,Reinbacher]
- Implemented in C++
- Parallelizable (MPI)
  Use 10 node dual-core Opteron cluster (Evelyn Thomson, ATLAS).

Testing the Result

How do we test whether the metric is the Calabi-Yau metric?
We could compute the Ricci tensor, but its easier to test that

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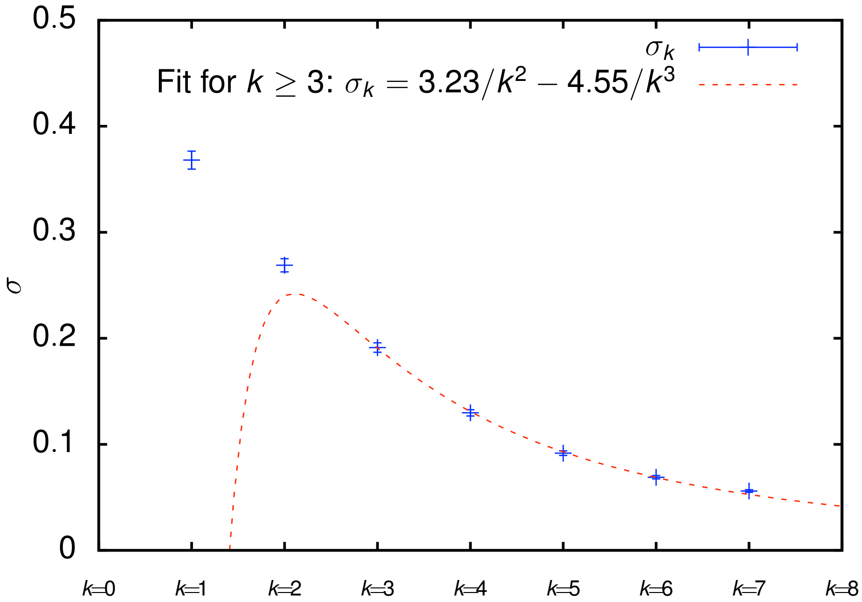
So normalize both volume forms and define

$$\sigma_k = \int_Q \left| 1 - \frac{\Omega(z) \wedge \bar{\Omega}(\bar{z})}{\omega^3(z, \bar{z})} \right| \mathrm{dVol}$$

On a Calabi-Yau manifold $\sigma_k = O(k^{-2})$


0.5
0.4
0.3
0.2
0.1
0
σ
σ_k
Fit for $k \geq 3$: $\sigma_k = 3.23/k^2 - 4.55/k^3$
k=0
k=1
k=2
k=3
k=4
k=5
k=6
k=7
k=8

Outline

1. **Introduction**
2. **Calabi-Yau Metrics**
3. **The $\mathbb{Z}_5 \times \mathbb{Z}_5$ Quotient**
   - Symmetric Quintics
   - Invariant Theory
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Symmetric Quintics

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- It is numerically much easier to work on the four-generation quotient $Q\Big/\big(\mathbb{Z}_5 \times \mathbb{Z}_5\big)$.

$$Q = \widetilde{Q}\Big/\big(\mathbb{Z}_5 \times \mathbb{Z}_5\big), \quad \mathcal{O}_Q(k) = \mathcal{O}_{\widetilde{Q}}(k)\Big/\big(\mathbb{Z}_5 \times \mathbb{Z}_5\big).$$

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- To do this, we only have to replace the sections s_α of $\mathcal{O}_{\widetilde{Q}}(k)$ by invariant sections!

$$H^0\big(Q, \mathcal{O}_Q(k)\big) = H^0\Big(\widetilde{Q}, \mathcal{O}_{\widetilde{Q}}(k)\Big)^{\mathbb{Z}_5 \times \mathbb{Z}_5}$$

Symmetry Group

$$g_1 \begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix}$$

$$g_2 \begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & e^{\frac{2\pi i}{5}} & 0 & 0 & 0 \\ 0 & 0 & e^{2\frac{2\pi i}{5}} & 0 & 0 \\ 0 & 0 & 0 & e^{3\frac{2\pi i}{5}} & 0 \\ 0 & 0 & 0 & 0 & e^{4\frac{2\pi i}{5}} \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix}$$

Note that $g_1 g_2 g_1^{-1} g_2^{-1} = e^{\frac{2\pi i}{5}}$, so they generate the Heisenberg group

$$0 \to \mathbb{Z}_5 \to G \to \mathbb{Z}_5 \times \mathbb{Z}_5 \to 0$$

Invariant Theory

The invariant sections are

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$$\mathbb{C}[z_0, z_1, z_2, z_3, z_4]^G = \bigoplus_{i=1}^{100} \eta_i \mathbb{C}[\theta_1, \theta_2, \theta_3, \theta_4, \theta_5]$$

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("Hironaka decomposition") where

$$\begin{aligned}
\theta_1 &\stackrel{\text{def}}{=} z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 \\
\theta_2 &\stackrel{\text{def}}{=} z_0 z_1 z_2 z_3 z_4 \\
\theta_3 &\stackrel{\text{def}}{=} z_0^3 z_1 z_4 + z_0 z_1^3 z_2 + z_0 z_3 z_4^3 + z_1 z_2^3 z_3 + z_2 z_3^3 z_4 \\
\theta_4 &\stackrel{\text{def}}{=} z_0^{10} + z_1^{10} + z_2^{10} + z_3^{10} + z_4^{10} \\
\theta_5 &\stackrel{\text{def}}{=} z_0^8 z_2 z_3 + z_0 z_1 z_3^8 + z_0 z_2^8 z_4 + z_1^8 z_3 z_4 + z_1 z_2 z_4^8
\end{aligned}$$

Secondary Invariants

... and the "secondary invariants" η_i are polynomials in degrees 0, 5, 10, 15, 20, 25, 30:

$$\eta_1 \stackrel{\text{def}}{=} 1$$

$$\eta_2 \stackrel{\text{def}}{=} z_0^2 z_1 z_2^2 + z_1^2 z_2 z_3^2 + z_2^2 z_3 z_4^2 + z_3^2 z_4 z_0^2 + z_4^2 z_0 z_1^2$$

$$\vdots$$

$$\eta_{100} \stackrel{\text{def}}{=} z_0^{30} + z_1^{30} + z_2^{30} + z_3^{30} + z_4^{30}$$

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- No invariant sections in $\mathcal{O}_{\tilde{Q}}(k)$ unless $5|k$?

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- All invariants are in degrees divisible by 5
- No invariant sections in $\mathcal{O}_{\tilde{Q}}(k)$ unless $5|k$?
- $\mathcal{O}_{\tilde{Q}}(k)$ only equivariant if $5|k$.

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   - Solving the Laplace Equation
   - Example: $\mathbb{P}^3$
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The Laplace-Beltrami Operator

The scalar Laplace operator

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In terms of some (non-orthogonal) basis of functions $\{f_s\}$, we can write

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(Generalized) eigenvalue equation

$$\begin{aligned} \Delta\big|\phi_i\big\rangle &= \lambda_i\big|\phi_i\big\rangle \\ \Rightarrow \quad \big\langle f_s\big|\Delta\big|f_t\big\rangle \underbrace{\big\langle f_t\big|\tilde{\phi}_i\big\rangle}_{\vec{v}} &= \lambda_i\big\langle f_s\big|f_t\big\rangle \underbrace{\big\langle f_t\big|\tilde{\phi}_i\big\rangle}_{\vec{v}} \end{aligned}$$

Spherical Harmonics

Using an approximate finite basis $\{f_s\}$, we only have to solve the generalized eigenvalue problem

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Nice basis: Recall that $\mathbb{P}^4 = S^9 \big/ U(1)$
So take the $U(1)$-invariant spherical harmonics on S^9.

Homogeneous Coordinates

In homogeneous coordinates, the spherical harmonics are

$$\frac{\Big(\text{degree } k \text{ monomial}\Big)\overline{\Big(\text{degree } k \text{ monomial}\Big)}}{\Big(|z_0|^2+|z_1|^2+|z_2|^2+|z_3|^2+|z_4|^2\Big)^k}$$

So, for example $k = 1$ on $\mathbb{P}^1$:

Homog.	$\frac{z_0\bar{z}_0}{\lvert z_0\rvert^2+\lvert z_1\rvert^2}$	$\frac{z_1\bar{z}_0}{\lvert z_0\rvert^2+\lvert z_1\rvert^2}$	$\frac{z_0\bar{z}_1}{\lvert z_0\rvert^2+\lvert z_1\rvert^2}$	$\frac{z_1\bar{z}_1}{\lvert z_0\rvert^2+\lvert z_1\rvert^2}$
Inhomog.	$\frac{1}{1+\lvert x\rvert^2}$	$\frac{x}{1+\lvert x\rvert^2}$	$\frac{\bar{x}}{1+\lvert x\rvert^2}$	$\frac{x\bar{x}}{1+\lvert x\rvert^2}$

Example: $\mathbb{P}^3$

Analytic result:

- Multiplicities of eigenvalues

$$\mu_n = \binom{n+3}{n}^2 - \binom{n+2}{n-1}^2, \quad n = 0, 1, \ldots$$

- Eigenvalues (normalize Vol $\mathbb{P}^3 = 1$)

$$\lambda_{n,0} = \cdots = \lambda_{n,\mu_n - 1} = \frac{4\pi}{\sqrt[3]{6}} n(n+3)$$

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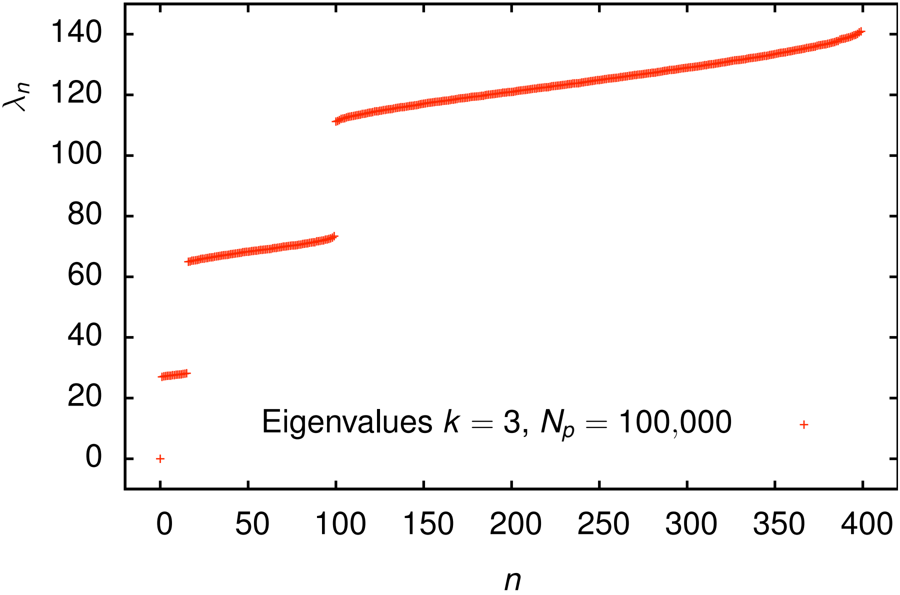
Numeric result: $k = 3$, $N_p = 100{,}000$.

Spectrum on $\mathbb{P}^3$: $k = 3$, $N_p = 100{,}000$



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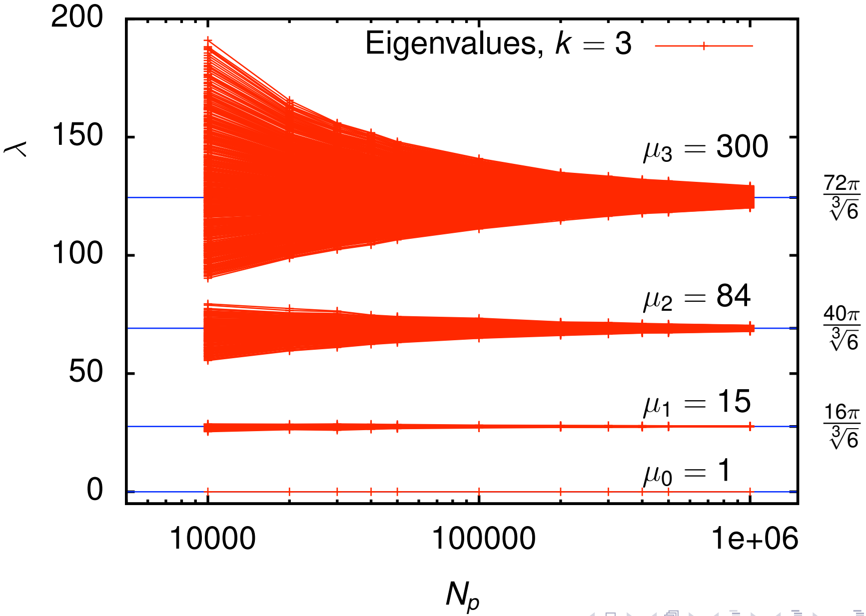
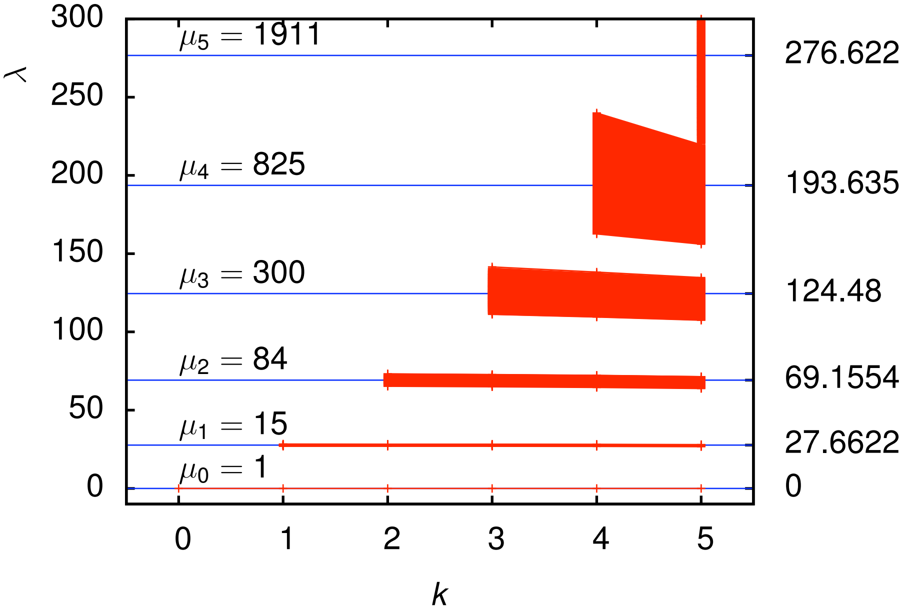


Spectrum on $\mathbb{P}^3$: $N_p = 100{,}000$



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   - Fermat Quintic
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Random Quintic

Now, take some quintic

$$Q(z) = (-0.3192 + 0.7096i)z_0^5 + (-0.3279 + 0.8119i)z_0^4 z_1 + (0.2422 + 0.2198i)z_0^4 z_2 + \cdots + (-0.2654 + 0.1222i)z_4^5$$

with 126 random (nonzero) coefficients.

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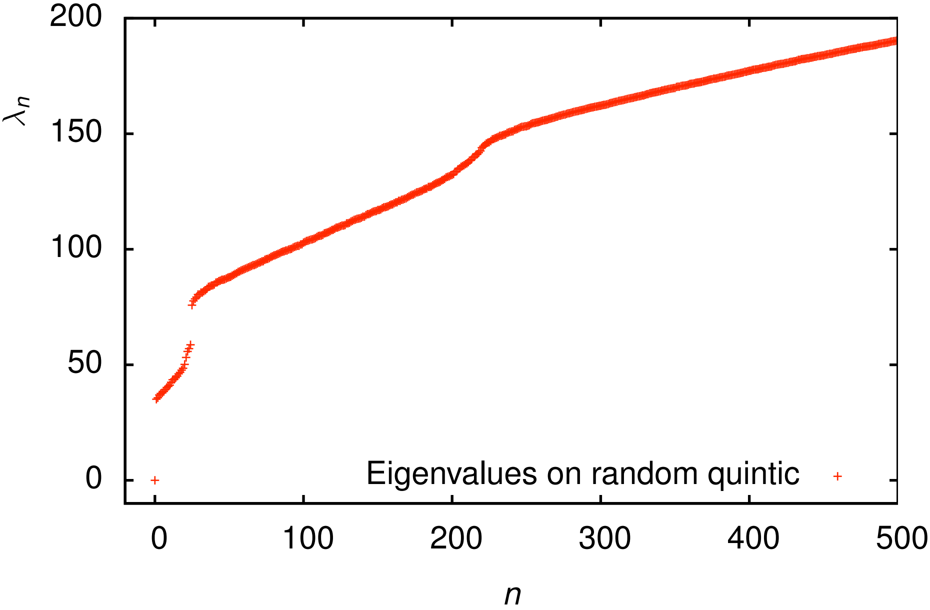
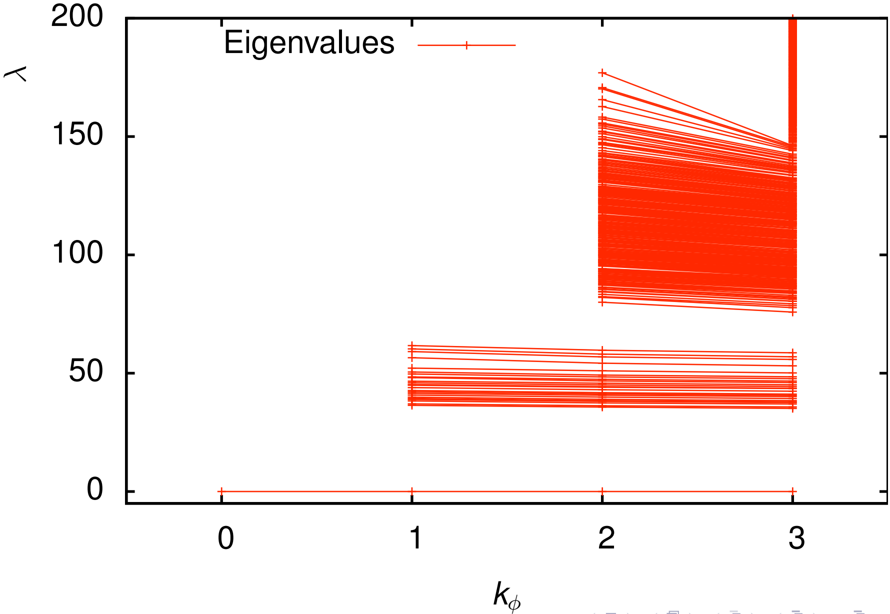
- Metric: $k_h = 8$.
- Integrate T-operator using 3,000,000 points.
- Normalize $\mathrm{Vol}(Q) = 1$.
- Laplacian: $k_\phi = 3$.
- Integrate using $N_p = 200{,}000$ points.

Random Quintic: $k_\phi = 3$, $N_p = 200{,}000$



Random Quintic: $N_p = 200{,}000$



Weyl's Formula

Theorem (Weyl)

$$\lim_{n\to\infty} \frac{\lambda_n^{d/2}}{n} = \frac{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}+1\right)}{\text{Vol}} \quad \left[= 384\pi^3 \right],$$

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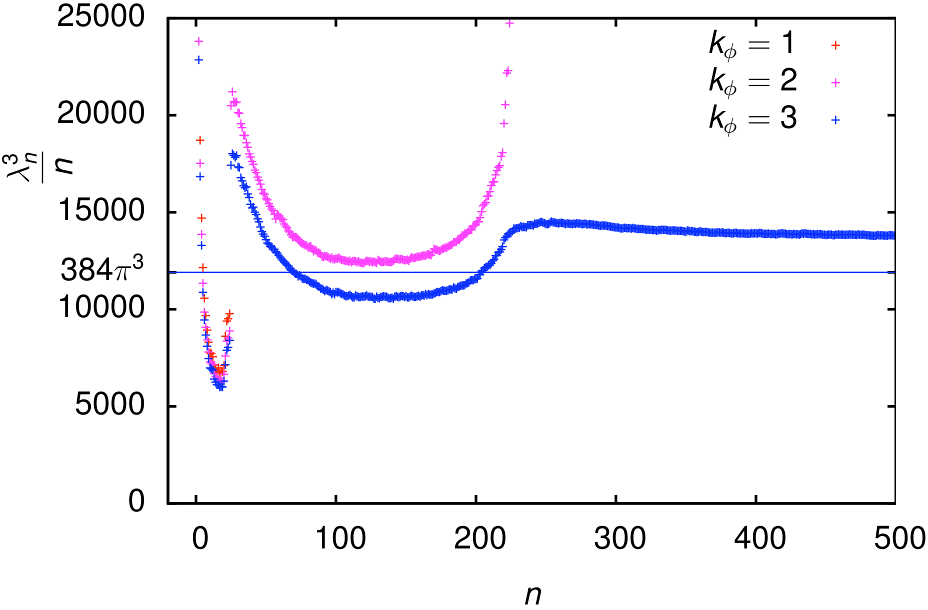
Independent check on the volume normalization.

Weyl's Limit



25000
20000
15000
$384\pi^3$
10000
5000
0
$\frac{\lambda_n^3}{n}$
0
100
200
300
400
500
n
$k_\phi = 1$
$k_\phi = 2$
$k_\phi = 3$

Massive Gravitons

Consider KK modes of the graviton that are spin-2 in 4 dimensions:

$$h_{\mu\nu}^{10d} = \sum_n h_{n,\mu\nu}^{4d}(x_0, x_1, x_2, x_3) \cdot \phi_n^{6d}(y_1, \dots, y_6)$$

Mass $m_n = \sqrt{\lambda_n}$.

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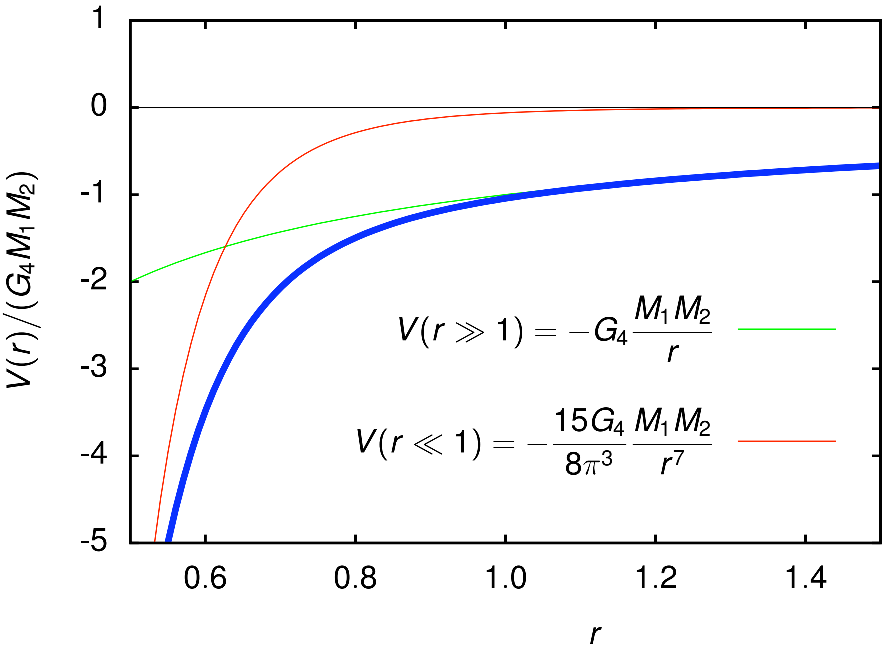
Mass $m_n = \sqrt{\lambda_n}$.

Gravitational potential between two test masses M_1 and M_2:

$$V(r) = -G_4 \frac{M_1 M_2}{r} \sum_{n=0}^{\infty} e^{-\sqrt{\lambda_n} r}$$



1
0
-1
-2
-3
-4
-5
0.6
0.8
1.0
1.2
1.4
r
$V(r)/(G_4 M_1 M_2)$
$V(r \gg 1) = -G_4 \frac{M_1 M_2}{r}$
$V(r \ll 1) = -\frac{15 G_4}{8\pi^3} \frac{M_1 M_2}{r^7}$

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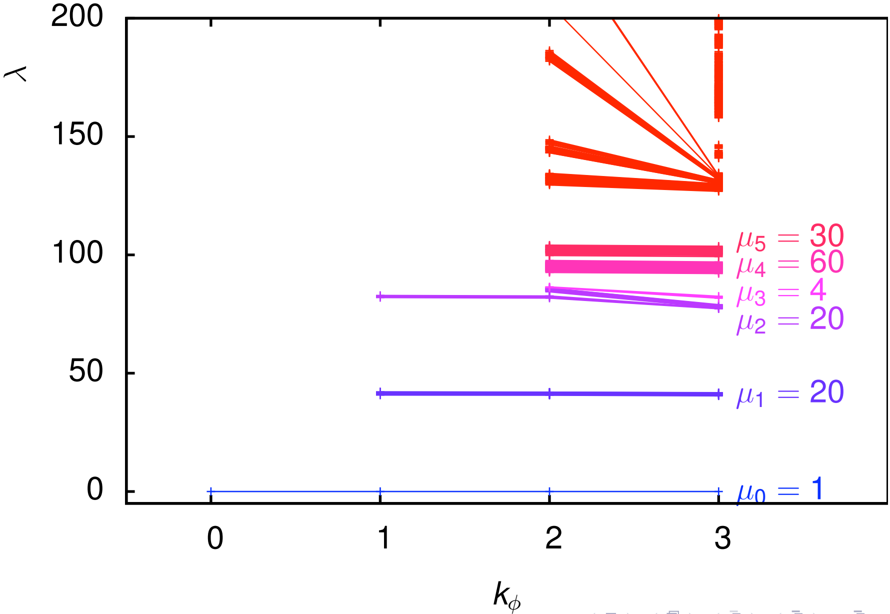
Fermat Quintic: $N_p = 500{,}000$



Symmetries of the Fermat Quintic

- The first massive eigenmode has degeneracy 20.

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- The automorphisms group of the Fermat quintic Q_F is

$$\overline{\mathrm{Aut}}\left(\widetilde{Q}_F\right) = \left(S_5 \times \mathbb{Z}_2\right) \ltimes \left(\mathbb{Z}_5\right)^4$$

Lemma (Representation theory of $\overline{\mathrm{Aut}}(\widetilde{Q}_F)$)

The 80 irreps of $\overline{\mathrm{Aut}}(\widetilde{Q}_F)$ are in dimension

Dimension d	1	2	4	5	6	8	10	…
Irreps in dim d	4	4	4	4	2	4	4	

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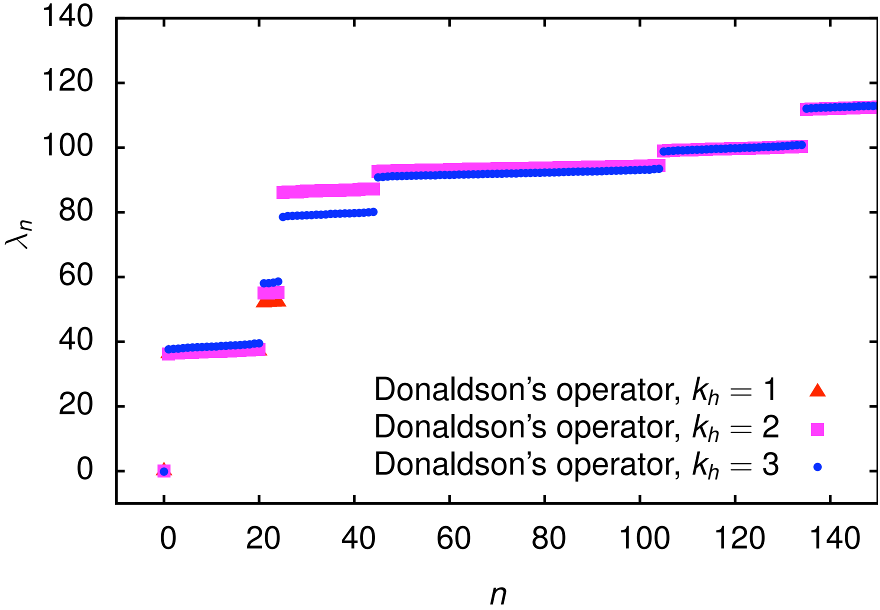
- Specific to the scalar Laplacian only.
- "Compares" balanced metrics at k and $2k$.

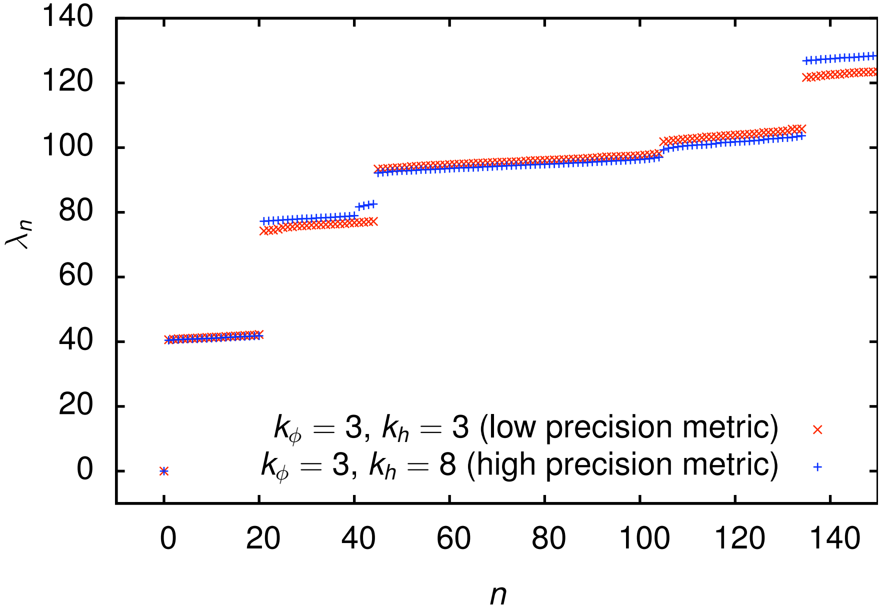
$$e^{\Delta} \sim Q_{\alpha\bar{\beta},\bar{\gamma}\delta} = \int (s_\alpha, s_\beta)\overline{(s_\gamma, s_\delta)}\, \mathrm{dVol}$$

$$\left[\text{Recall:} \quad T(h)_{\alpha\bar{\beta}} = \int (s_\alpha, s_\beta)\, \mathrm{dVol} \right]$$

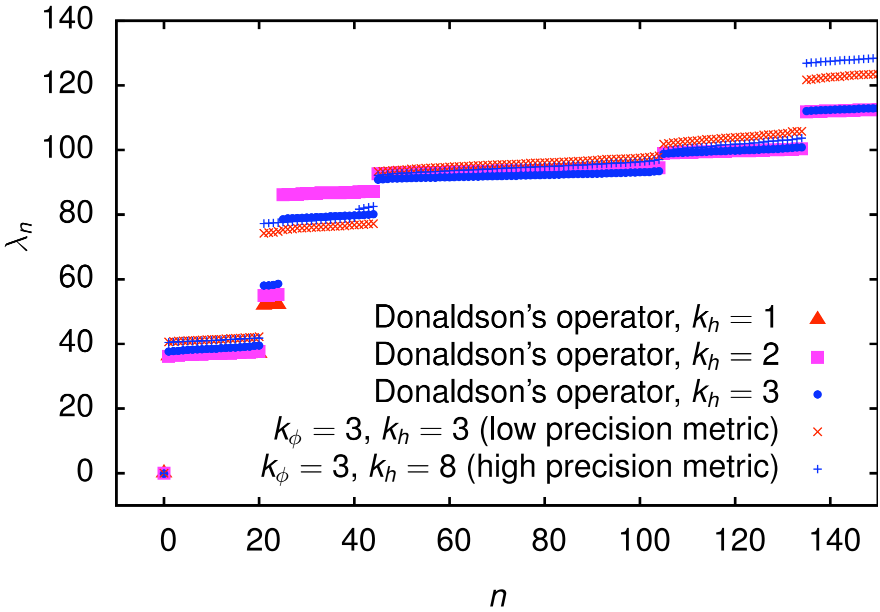
Fermat Quintic: Donaldson's Operator



Fermat Quintic: scalar Laplacian



Donaldson vs. scalar Laplacian



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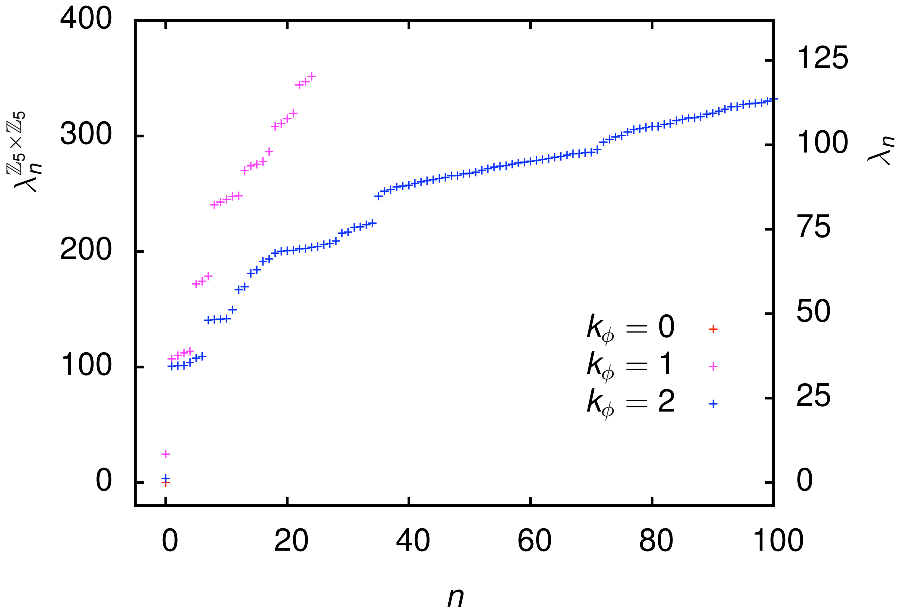
- Yields eigenvalues $\lambda_n^{\mathbb{Z}_5 \times \mathbb{Z}_5}$.
- Rescale volume to one:

$$\frac{1}{25} = \mathrm{Vol}\left(Q_F\right) \longrightarrow 1$$

$$\lambda_n^{\mathbb{Z}_5 \times \mathbb{Z}_5} \longrightarrow 25^{-1/3} \lambda_n^{\mathbb{Z}_5 \times \mathbb{Z}_5} = \lambda_n$$

Fermat Quintic $/(\mathbb{Z}_5 \times \mathbb{Z}_5)$: $N_p = 100{,}000$



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Quintic$\big/(\mathbb{Z}_5 \times \mathbb{Z}_5)$ Family #1

A family of Quintics

$$\widetilde{Q}_\psi = \left\{ \sum z_i^5 - 5\psi \prod z_i = 0 \right\}$$

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- $\psi \to \infty$ is the “large complex structure limit”.

Conifold Point of the Quintic

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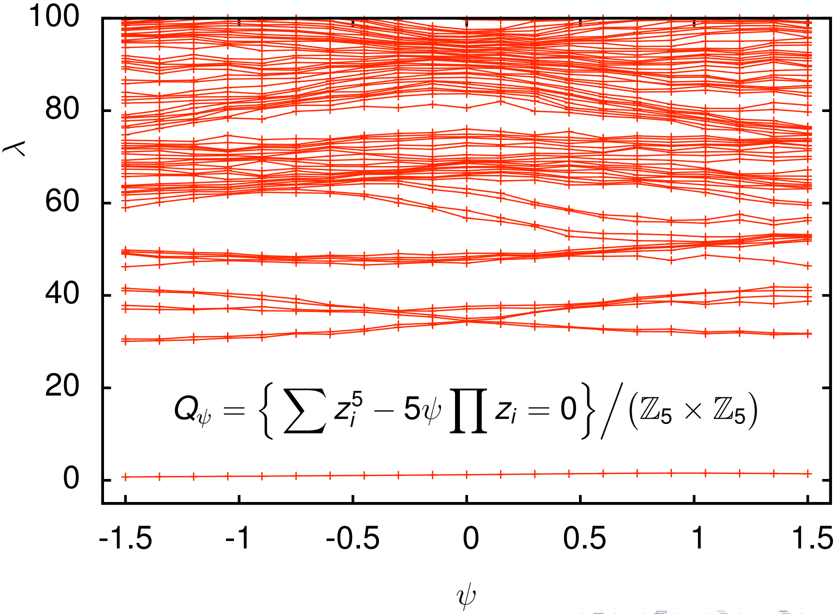
$$z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 - 5 z_0 z_1 z_2 z_3 z_4 = 0$$

is singular at $z_C = [1 : 1 : 1 : 1 : 1]$:

$$Q_1(z_C) = 0 = \frac{\partial Q_1}{\partial z_i}(z_C)$$

Quintic$/(\mathbb{Z}_5 \times \mathbb{Z}_5)$ Family #1



Large Complex Structure Limit

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Any Calabi-Yau threefold is fibered by special Lagrangian 3-tori over a (rational homotopy) 3-sphere. In the large complex structure limit the 3-torus fiber shrinks to zero size.

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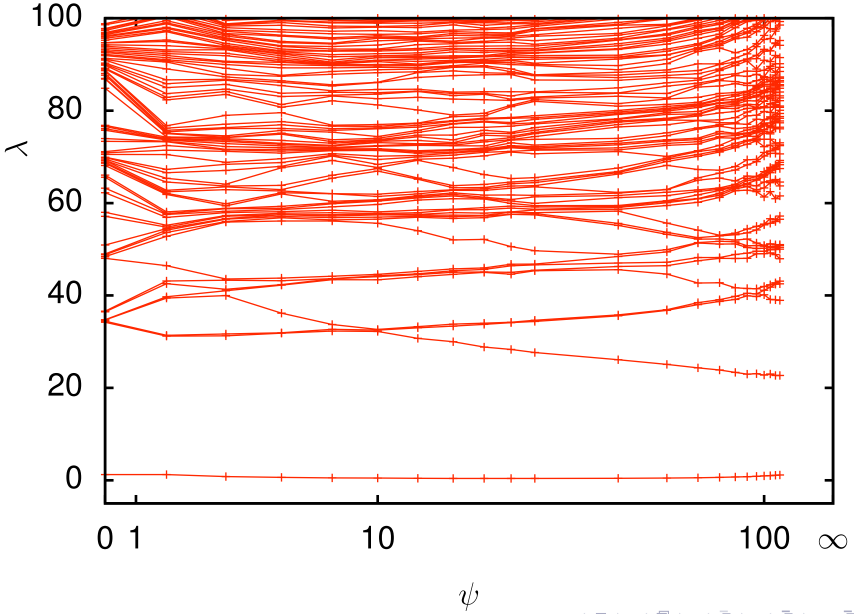
Hence, the spectrum of the Laplacian degenerates into the spectrum of the base space.

Large Complex Structure Limit



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Another family of Quintics

$$
\begin{aligned}
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& + (1-i)\varphi\big(z_0^2 z_1 z_2^2 + \text{cyc}\big) - (1-2i)\varphi\big(z_0^2 z_1^2 z_3 + \text{cyc}\big) \\
& - (2-i)\varphi\big(z_0^3 z_2 z_3 + \text{cyc}\big)
\end{aligned}
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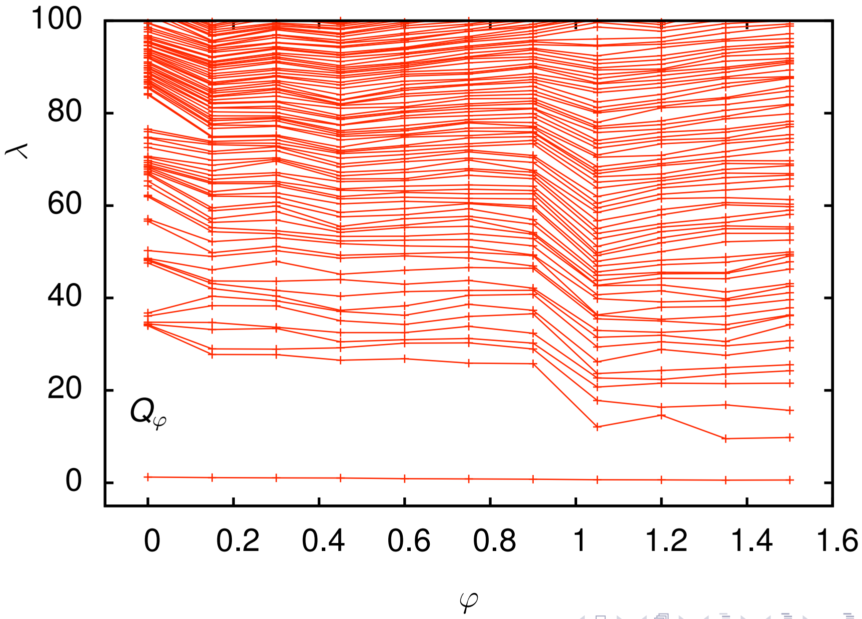
- $\widetilde{Q}_0$ is again the Fermat quintic.

Quintic $/(\mathbb{Z}_5 \times \mathbb{Z}_5)$ Family #2



Spectral Gap

Definition

The spectral gap is the gap between the unique zero mode $\lambda_0 = 0$ and λ_1.

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Theorem

On a non-negatively curved manifold (e.g. Calabi-Yau)

$$\frac{\pi^2}{D^2} \leq \lambda_1 \leq \frac{2d(d+4)}{D^2},$$

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Essentially determined by diameter!

The Diameter

Turn inequality around and estimate diameter from the spectral gap.

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Fermat quintic has $\lambda_1 \approx 41.1 \Rightarrow$

$$0.490 \approx \frac{\pi}{\sqrt{\lambda_1}} \leq D \leq \frac{\sqrt{2 \cdot 6(6+4)}}{\sqrt{\lambda_1}} \approx 1.71$$

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The Laplace-Dolbeault Operator

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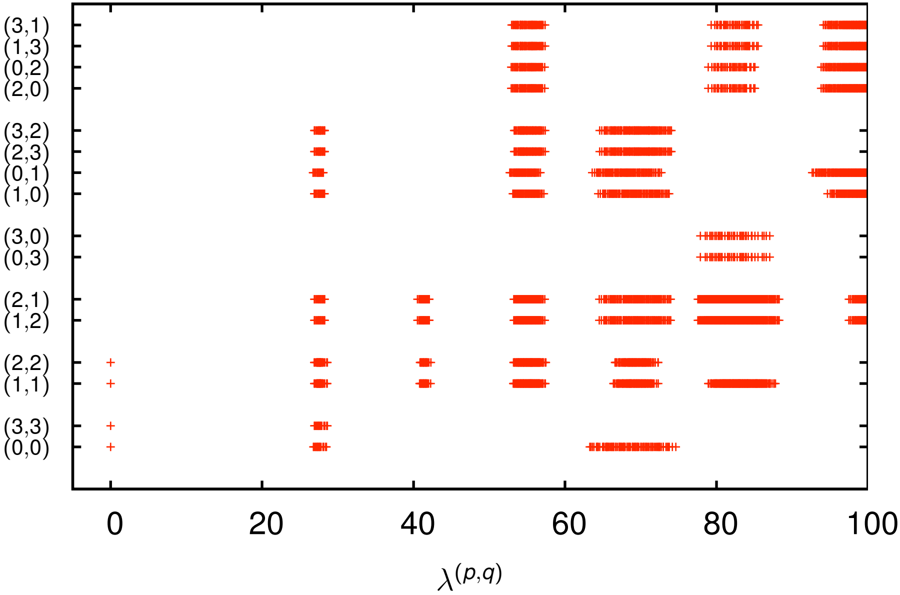
Complex conjugation & Hodge star:

$$\lambda_n^{(p,q)} = \lambda_n^{(q,p)} = \lambda_n^{(3-p,3-q)} = \lambda_n^{(3-q,3-p)}$$

Differential Forms on $\mathbb{P}^3$



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