Special Geometry, Black Holes and Instantons

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Outline

This talk is about:

- the special geometry of $d = 4$ $n = 2$ vector multiplets for both Lorentzian and Euclidean space-time signature
- and its application to black holes and instantons.

Try to give a broad overview of the topic. My own contributions were/are made in collaboration with Klaus Behrndt, Gabriel Lopes Cardoso, Vicente Cortés, Bernard de Wit, Renata Kallosh, Jürg Käppeli, Dieter Lüst, Christoph Mayer,Frank Saueressig, Ulrich Theis, Kirk Waite.

For references see `hep-th/0703035, hep-th/0703037 and to appear.`

Special Geometry

Special holonomy and related special geometric structures in string theory

1. Geometry of space-time.
2. Geometry of compact additional dimensions ('compactification').
3. Geometry of target spaces of sigma models. Often the 'moduli spaces' arising in compactification.

We will discuss aspects of point 3 (special geometry of sigma model target spaces), and its interplay with points 1,2 (black hole and instanton solutions of effective field theories arising form 'string compactifications').

Special geometry, Lorentzian space time

Sigma model (plus gravity)

Action:

$$S[\phi] \simeq \int d^4x \sqrt{|g|} g^{\mu\nu} G_{ab}(\phi) \partial_\mu \phi^a \partial_\nu \phi^b$$

Scalars ϕ^a = components of a map

$$\phi : (S, g) \longrightarrow (M, G)$$

from space-time (S, g) to target space (M, G), both (pseudo-)Riemannian.
Critical points of $S[\phi]$ correspond to harmonic maps:

$$\Delta_{(g)} \phi^a + \Gamma^a_{bc}(G) g^{\mu\nu} \partial_\mu \phi^b \partial_\nu \phi^c = 0 \, .$$

$N = 1$ supersymmetric Sigma model

Complex scalars $\subset$ Chiral multiplets (z, λ).
(M, G) is (pseudo-)Kähler.

$$S \simeq \int d^4x \sqrt{|g|} g^{\mu\nu} G_{i\bar{j}}(z) \partial_\mu z^i \partial_\nu \overline{z^j} + \cdots$$

We left out fermions and auxiliary fields. The space-time metric may be a background or dynamical (add Einstein-Hilbert term).

$N = 2$ supersymmetric Sigma models

$N = 2$ vector multiplets: $(X^I, \lambda^{Ii}, A^I_\mu)$
(plus auxiliary fields when considering off-shell version).
$I = 1, \ldots, n$ labels the vector multiplets, $i = 1, 2$.

Gauge field sector: field equations invariant under electric-magnetic duality rotations.

$$\begin{pmatrix} F^{I|\pm}_{\mu\nu} \\ G^{\pm}_{I|\mu\nu} \end{pmatrix} \longleftarrow Sp(2n, \mathbb{R})$$

(suppressed additional affine transformation present in rigid case.)

Field strength: $F^I_{\mu\nu} = \partial_\mu A^I_\nu - \partial_\nu A^I_\mu$.
Dual field strength:

$$G^{\pm}_{I|\mu\nu} = \frac{\delta \mathcal{L}}{\delta F^{I|\pm|\mu\nu}}$$

'$\pm$' = (anti-)selfdual part. $\mathcal{L} =$ Lagrangian.

$N = 2$ supersymmetric Sigma models

Scalars X^I must also be part of a 'symplectic vector':

$$\begin{pmatrix} X^I \\ F_I \end{pmatrix} \longleftarrow Sp(2n, \mathbb{R}) \,.$$

F_I are dependent quantities: $F_I = F_I(X)$.
In a generic symplectic frame

$$F_I(X) = \frac{\partial F(X)}{\partial X^I} \,,$$

where $F(X)$ is a holomorphic function, the prepotential, which encodes all couplings of the vector multiplet Lagrangian.
Scalar target space M is (affine) special Kähler.

Geometry of the prepotential

Geometrical interpretation: there exists a Kählerian Lagrangian immersion

$$\Phi = dF : M \longrightarrow T^*\mathbb{C}^n \,.$$

Equivalent to the intrinsic definition of affine special Kähler manifolds: Kähler $\oplus$ existence of a flat, torsion free, symplectic connection satisfying $\nabla_X I(Y) = \nabla_Y I(X)$.

(X^I, F_I) coordinates on $T^*\mathbb{C}^n$.
$M \to \Phi(M) \subset T^*\mathbb{C}^n$ is locally a complex Lagrangian submanifold $X^I \to F_I(X)$.

Off-shell construction using the superconformal calculus.

- Take matter multiplets with rigid superconformal symmetry.
- ‘Gauge’ superconformal symmetry.
- Impose ‘gauge conditions’ which leave Poincaré supersymmetry intact but fix the additional superconformal symmetries.

Remark: gravitational degrees of freedom encoded in superconformal connections (Weyl multiplet).

- Rigid superconformal invariance $\Leftrightarrow$ prepotential is homogenous of degree 2. Scalar target space M is complex cone.
- Gauging superconformal symmetry = coupling to Weyl multiplet. 'Gauge equivalence' with Poincaré supergravity requires the following field content:

$$\text{Conf. Sugra} = \text{Weyl} \oplus (n+1)\text{vector multiplets} \oplus 1\text{hypermultiplet}$$

- Upon gauge fixing obtain:

$$\text{Poincaré Sugra} = \text{gravity multiplet} \oplus n \text{ vector multiplets}$$

1 vector multiplet and 1 hypermultiplet act as 'compensators'. Number of gauge fields $F^I_{\mu\nu}$ unchanged: one gauge field ('graviphoton') sits in the gravity multiplet.

Projective special Kähler geometry

Gauge fixing of complex dilational symmetry

$$X^I \longrightarrow e^{w - ic} X^I$$

reduces the number of complex scalar fields by one.
Physical scalars can be taken to be

$$z^i = \frac{X^i}{X^0} \; , \quad i = 1, \ldots, n \; .$$

and parametrize a projective special Kähler manifold $\overline{M}$.

Projective special Kähler geometry

Complex dilatation gauge symmetry = $\mathbb{C}^*$-action on the complex cone M.

$\overline{M}$ is obtained from M by taking a Kähler quotient:

$$\overline{M} = M/\mathbb{C}^* \; .$$

'Using the gauge equivalence between conformal and Poinaré supergravity' $\leftrightarrow$ analyzing $\overline{M}$ in terms of M.
This allows to keep symplectic covariance manifest!
NB: string dualities (S-duality, T-duality, monodromy group of prepotential) operate by symplectic transformations.

Projective special Kähler geometry

Poincaré SuGra n vector mult.	$\longleftrightarrow$	Conf. SuGra $(n+1)$ vector mult.		
$\overline{M}$	$\longleftrightarrow$	M	$\stackrel{\Phi}{\longrightarrow}$	$T^*\mathbb{C}^{n+1}$
z^i		X^I		$\begin{pmatrix} X^I \\ F_I \end{pmatrix}$

Black Holes

$\frac{1}{2}$-BPS Black Holes

Application: $\frac{1}{2}$-BPS solutions of $N = 2$ supergravity $\oplus$ vector multiplets.

- Relevant part of the 4d low energy effective field theory of string compactifications type-II/Calabi-Yau threefold, heterotic/K3 $\times$ two-torus.
- $\frac{1}{2}$-BPS: 4 (physical) Killing spinors (out of maximal 8).
- Restrict here to static, spherically symmetric solutions = non-rotating black holes. (Generalisations: Rotating black holes, multi-black hole solutions.)
- Automatically extremal: $T_{\rm Hawking} = 0$, $M = |Z|$.
  M=Mass, Z=central charge of $N = 2$ algebra.

Symplectic covariance

Symplectic vectors:

$$\begin{pmatrix} X^I \\ F_I \end{pmatrix}, \quad \begin{pmatrix} F^I_{\mu\nu} \\ G_{I|\mu\nu} \end{pmatrix} \xrightarrow{\oint} \begin{pmatrix} p^I \\ q_I \end{pmatrix}.$$

p^I=magnetic charges, q_I = electric charges.

Symplectic scalars:

- Graviphoton: $\mathcal{F}^-_{\mu\nu} \simeq X^I G^-_{I|\mu\nu} - F_I F^{I|-}_{\mu\nu}$.
- Central charge: $Z \simeq \oint \mathcal{F}^- \simeq (p^I F_I - q_I X^I)_{|\infty}$.
- 'Central charge': $Z = p^I F_I - q_I X^I$.
- The prepotential F is not a symplectic scalar (but $F - \frac{1}{2} X^I F_I$ is).

Black Hole Solutions

Solution reduces to 'attractor equations'for the scalars (algebraic version, equivalent to gradient flow equations for the z^i).

$$\begin{pmatrix} \mathcal{X}^I - \overline{\mathcal{X}}^I \\ F_I - \overline{F}_I \end{pmatrix} = i \begin{pmatrix} H^I \\ H_I \end{pmatrix}$$

where

- $\mathcal{X}^I \propto X^I$ are the (uniformly rescaled) scalars on M.
  Note: F_I is homogenous of degree 1.
- H^I, H_I are harmonic functions on $\mathbb{R}^3$.
  Spherically symmetric 'single centered' case:

$$H^I = h^I + \frac{p^I}{r}, \quad H_I = h_I + \frac{q_I}{r}.$$

Black Hole Solutions

Metric and gauge field determined by scalars

- Metric (conforma-static form)

$$ds^2 = -e^{-2f(r)}dt^2 + e^{2f(r)}(dr^2 + r^2 d\Omega^2)$$

  where

$$e^{2f(r)} = i\left(\overline{\mathcal{X}}^I F_I - \overline{F}_I \mathcal{X}^I\right)$$

- Gauge fields are determined by magneto-static and electro-static potentials

$$\begin{pmatrix} \phi^I \\ \chi_I \end{pmatrix} \propto \begin{pmatrix} \mathcal{X}^I + \overline{\mathcal{X}}^I \\ F_I + \overline{F}_I \end{pmatrix} .$$

Horizon limit

Attractor mechanism: at the horizon ($r \to 0$) the solutions is completely determined by the charges and becomes independent of asymptotic moduli $z^i_\infty \leftrightarrow h^I, h_I$.

- Attractor values of scalars:

$$\left(\begin{array}{c} Y^I - \overline{Y}^I \\ F_I - \overline{F}_I \end{array} \right)_{|*} = i \left(\begin{array}{c} p^I \\ q_I \end{array} \right) ,$$

  where $Y^I \propto X^I$ are the (uniformly rescaled) scalars on M.
- Metric is asymptotic to $AdS^2 \times S^2$

$$ds^2 = -\frac{r^2}{|Z_*|^2} dt^2 + \frac{|Z_*|^2}{r^2} dr^2 + |Z_*|^2 d\Omega^2 .$$

- Solutions becomes maximally supersymmetric (8 Killing spinors).

Bekenstein-Hawking entropy (symplectic scalar):

$$S_{\rm BH} = \frac{A}{4} = \pi |Z_*|^2 = \pi \left(p^I F_I - q_I Y^I \right)_* ,$$

$A =$ horizon area.

Variational Principle

Define:

- Entropy function:
  $\Sigma(Y, \overline{Y}, p, q) = \mathcal{F}(Y, \overline{Y}) - q_I(Y^I + \overline{Y}^I) + p^I(F_I + \overline{F}_I)$.
- Free energy: $\mathcal{F}(Y, \overline{Y}) = -i(\overline{Y}^I F_I - Y^I \overline{F}_I)$.

Then

- Crtical points of Σ with respect to Y^I = attractor points.
- Critical value of Σ = Entropy

$$\pi \Sigma_* = \pi |Z_*|^2 = S_{BH}(p, q) \; .$$

Entropy = Legendre transf. of Hesse potential

Use special affine coordinates on M:

$$\begin{pmatrix} x^I \\ y_I \end{pmatrix} = \mathrm{Re} \begin{pmatrix} Y^I \\ F_I \end{pmatrix} \propto \begin{pmatrix} \phi^I \\ \chi_I \end{pmatrix}$$

(affine coordinates of the special connection on M).

- Free energy $\propto$ Hesse potential $H(x,y) = H(\phi,\chi)$.
- Entropy = Legendre transform of Hesse potential

$$S_{\mathrm{BH}}(p,q) = 2\pi \left(H - x^I \frac{\partial H}{\partial x^I} - y_I \frac{\partial H}{\partial y_I} \right)_{|*}$$

$$\frac{\partial H}{\partial x^I} = q_I \,, \quad \frac{\partial H}{\partial y^I} = -p_I \,.$$

Mixed ensemble

Reduced variational principle: Solve magnetic attractor equation $Y^I - \overline{Y}^I = ip^I$ by

$$Y^I = \frac{1}{2}\left(\phi^I + ip^I\right)$$

to obtain the 'mixed' entropy function

$$\Sigma_{\text{mix}}(\phi, p, q) = \mathcal{F}_{\text{mix}}(p, \phi) - q_I \phi^I$$

and the 'mixed' free energy

$$\mathcal{F}_{\text{mix}}(p, \phi) = 4\text{Im}F(Y(p, \phi)) \; .$$

Entropy = partial Legendre transform of $\mathcal{F}_{\text{mix}}$ wrt ϕ^I.
This was used to formulate the 'OSV-conjecture.'
NB: $\mathcal{F}_{\text{mix}}$ is not a symplectic function and p^I, ϕ^I do not form a symplectic vector.

R^2-corrections

String theory predicts higher derivative corrections to the effective action, such as

$$\mathcal{F}^{(h)}(z^i)(\text{Riemann})^2(\text{Gauge Fd.})^{2h-2}\;, \quad h \geq 1\;,$$

which modify black hole solutions and black hole entropy. These corrections can be calculated, and have successfully been matched with the statistical entropy obtained by counting microstates. Here we restrict ourselves to the 'macroscopic' aspects.

R^2-corrections

- One particular class of higher derivative terms ('R^2-terms'), is captured by giving the prepotential an explicit dependence on the Weyl multiplet:

$$F(Y^I) \to F(Y^I, \Upsilon) = \sum_{h=0}^{\infty} F^{(h)}(Y^I)\Upsilon^h$$

(graded homogenous of degree 2, Υ has weight 2).

- Precisely this class of terms is encoded in the topological string: $F^{(h)}(Y^I) \leftrightarrow \mathcal{F}^{(h)}(z^i) =$ topological free energy (genus h).

R^2-corrected black hole solutions

Using the superconformal off-shell formulation, 'everything' (attractor mechanism, variational principle, global solutions,...) can be generalised to include R^2-corrections.

- Attractor equations:

$$\left(\begin{array}{c} Y^I - \overline{Y}^I \\ F_I(Y, \Upsilon) - \overline{F}_I(\overline{Y}, \overline{\Upsilon}) \end{array} \right)_{|*} = i \left(\begin{array}{c} p^I \\ q_I \end{array} \right) , \quad \Upsilon_* = -64 .$$

- Entropy (symplectic scalar):

$$S_{\rm Wald}(p,q) = \pi \Sigma_* = \pi \left[p^I F_I(Y, \Upsilon) - q_I Y^I + 4 \mathrm{Im} \left(\Upsilon \frac{\partial F}{\partial \Upsilon} \right) \right]_{|*}$$

NB: Entropy $\neq \frac{1}{4}$ Area. Essential for matching statistical entropy!

OSV conjecture

Observation: mixed free energy is (essentially) the all-genus topological free energy

$$\exp\left(\pi \mathcal{F}_{\rm mix}(p,\phi)\right) = \exp\left(2{\rm Re} F_{\rm top}(p,\phi)\right) = |Z_{\rm top}|^2$$

Conjecture: the left hand side is the ('mixed') partition function counting black hole microstates:

$$Z_{\rm mix}(p,\phi) := \sum_q d(p,q) e^{q_I \phi^I}$$

i.e.

$$Z_{\rm mix}(p,\phi) \stackrel{\rm OSV}{=} |Z_{\rm top}(p,\phi)|^2 \; .$$

OSV conjecture

$$Z_{\mathrm{mix}}(p,\phi) \stackrel{\mathrm{OSV}}{=} |Z_{\mathrm{top}}(p,\phi)|^2 \ ?$$

- True to leading order for large charges courtesy the variational principle.
- Cannot be exact, because in contradiction to symplectic covariance and, hence, duality invariance. Main problem: incorporation of subleading non-holomorphic corrections.
- Open: what is the correct modification? Is the resulting statement exact or asymptotic, and if the latter, where do deviations start?
- The presence of a 'measure factor' which corrects the OSV formula has been demonstrated in several examples. We have made a 'minimal' proposal based on imposing symplectic covariance, which is correct (so far) within the semiclassical (saddle point) approximation.

Non-holomorphic corrections

'Wilsonian' couplings encoded in holomorphic $F(Y^I, \Upsilon) \neq$ 'physical' (duality invariant) couplings.

Example: coefficient of (Weyl tensor)^2 in $N = 4$ compactifications

$$\Omega_{R^2} \propto \log \eta^{24}(iS) + \overline{\log \eta^{24}(iS)} + \log(S + \overline{S})^{12} \neq \mathrm{Im}(F^{(1)}_{\mathrm{hol}}(S))$$

Dilaton S transforms as

$$S \to \frac{aS + ib}{-icS + d}$$

under S-duality $SL(2, \mathbb{Z})_S$.

Non-holomorphic corrections

Non-holomorphic corrections can be incorporated systematically in the variational principle, attractor equations, and entropy. I.p. attractor equations take symplectically covariant form

$$\left(\begin{array}{c} Y^I - \overline{Y}^I \\ F_I(Y, \Upsilon) + 2i\Omega_I - \overline{F}_I(\overline{Y}, \overline{\Upsilon}) + 2i\Omega_{\bar{I}} \end{array} \right)_{|*} = i \left(\begin{array}{c} p^I \\ q_I \end{array} \right) , \quad \Upsilon_* = -64 .$$

where $\Omega(Y, \overline{Y}, \Upsilon, \overline{\Upsilon})$ is a real-valued, homogenous function (i.g. not harmonic).

Integrate to 'non-holomorphic prepotential'?

Modified OSV conjecture

- Use 'canonical' ensemble, rather than 'mixed ensemble'. Free energy $\mathcal{F}(\phi, \chi, \Upsilon) =$ (generalised) Hessepotential.
- Free energy includes 'non-holomorphic' corrections through non-harmonic Ω.
- Conjecture: canonical free energy related to canonical black hole partition function by

$$e^{\pi \mathcal{F}(\phi,\chi)} \approx Z_{\mathrm{can}}(\phi, \chi) := \sum_{p,q} d(p,q) e^{\pi(q_I \phi^I - p^I \chi_I)} \; .$$

- Equivalent to modifying the OSV formula by a specific measure factor:

$$Z_{\mathrm{mix}}(p, \phi) = \sqrt{\Delta^-} |Z_{\mathrm{top}}|^2 \; .$$

  Proposal works in saddle point approximation including subleading corrections.

Non-holomorphic corrections (again)

Major technical complication

- Superconformal formalism uses full $(Y^I, F_I(Y, \Upsilon) + 2i\Omega(Y, \overline{Y}, \Upsilon, \overline{\Upsilon}))$.
  Non-holomorphic corrections encoded in non-harmonic Ω.
- Topological string uses expanded version

$$F_{\text{top}}(z^i, g_{\text{top}}) = \sum_{h=0}^{\infty} g_{\text{top}}^{2h-2} F_{\text{top}}^{(h)}(z^i)$$

  Monodromy properties of $F_{\text{top}}^{(h)}(z^i)$.
  Non-holomorphic corrections: holomorphic anomaly equations.

  Note $\Upsilon =$ const. at horizon, and

$$F^{(h)}(Y) = (Y^0)^{2-2h} F^{(h)}\left(\frac{Y^i}{Y^0}\right) \propto g_{\text{top}}^{2h-2} F_{\text{top}}^{(h)}(z^i)$$

Both formalism encode non-holomorphic corrections in very different ways!

Euclidean space, split target

Why Euclidean?

Why consider Euclidean 'space-time' S?

1. Quantum mechanics and Quantum field theory: path integral/ functional integral (better) defined.
2. Quantum tunneling, Instantons $\leftrightarrow$ classical solutions in 'imaginary time' (non-trival saddle points of the Euclidean path/functional integral).
3. Soliton/Instanton correspondence: Stationary solution in $d+1$ (Lorentzian) dimensions $\leftrightarrow$ Solution in d (Euclidean) dimensions.

   Moreover, treating time (or part of space) as 'internal' reveals hidden symmetries (aka U-dualities).

How to relate Lorentzian and Euclidean theories?

1. QM and QFT: 'Wick rotation' $t \to -it$. Scalar field space M not modified.
2. Lift and reduce: $3+1 \to 4+1 \to 4+0$.
   Not always applicable, but natural in the soliton/instanton connection. Geometry of scalar field space different from original M.

Both methods give different Euclidean actions, which might be viewed as different 'real forms' of a complex action.
Thus 'type 2' can be defined without reference to dimensional lifting/reduction, using analytic continuation in field space.
Alternatively: use that Wick rotation and Hodge dualisation do not commute.

We focus on geometrical aspects in the following.

Toy example

One real scalar, one gauge field in $4+1$ dimensions, reduce to $3+1$ ($\epsilon = -1$) and $4+0$ ($\epsilon = 1$):

- Five dimensions:

$$\mathcal{L} = -\partial_\mu \sigma \partial^\mu \sigma - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

- Four dimensions:

$$\mathcal{L}_\epsilon = -(\partial_m \sigma \partial^m \sigma - (-\epsilon)\partial_m b \partial^m b) - \frac{1}{4} F_{mn} F^{mn}$$

  where $b \simeq A_4$ for spatial reduction ($\epsilon = -1$) and $b \simeq A_0$ for temporal reduction ($\epsilon = 1$).

Minkowski space $S_{3,1}$ ($\epsilon = -1$): positive definit target space metric.
Euclidean space S_4 ($\epsilon = 1$): split signature target space metric.
Lagrangians are related by Wick rotation $t \to -it$ combined with analytical continuation $b \to ib$.

Focus on scalar part of 4d Lagrangian:

- $$\mathcal{L}_{(-1)} = -(\partial_m \sigma \partial^m \sigma + \partial_m b \partial^m b) + \cdots = -\partial_m z \partial^m \overline{z} + \cdots$$
  where $z = \sigma + ib$. Target space geometry is complex.
- $$\mathcal{L}_{(1)} = -(\partial_m \sigma \partial^m \sigma - \partial_m b \partial^m b) + \cdots = -\partial_m z_+ \partial^m z_- + \cdots$$
  where $z_+ = \sigma + b$, $z_- = \sigma - b$. Light cone coordinates. But we can do better!

ϵ-complex geometry

Focus on scalar part of 4d Lagrangian:

- $$\mathcal{L}_{(-1)} = -(\partial_m \sigma \partial^m \sigma + \partial_m b \partial^m b) + \cdots = -\partial_m z \partial^m \overline{z} + \cdots$$

  where $z = \sigma + ib$. Target space geometry is complex.

- $$\mathcal{L}_{(1)} = -(\partial_m \sigma \partial^m \sigma - \partial_m b \partial^m b) + \cdots = -\partial_m z \partial^m \overline{z} + \cdots$$

  where $z = \sigma + eb$, with para-complex unit e:
  $e^2 = 1$, $\overline{e} = -e$. Target space geometry is para-complex.

ϵ-complex geometry

Focus on scalar part of 4d Lagrangian:

- Uniform description, in terms of ϵ-complex geometry.

$$\mathcal{L}_{(\epsilon)} = -(\partial_m \sigma \partial^m \sigma - (-\epsilon)\partial_m b \partial^m b) + \cdots = -\partial_m z \partial^m \overline{z} + \cdots$$

where $z = \sigma + i_\epsilon b$ and

$$i_\epsilon = \begin{cases} i & \text{for } \epsilon = -1 \\ e & \text{for } \epsilon = 1 \end{cases}$$

ϵ-complex geometry

Focus on scalar part of 4d Lagrangian:

- Uniform description, in terms of ϵ-complex geometry.

$$\mathcal{L}_{(\epsilon)} = -(\partial_m \sigma \partial^m \sigma - (-\epsilon)\partial_m b \partial^m b) + \cdots = -\partial_m z \partial^m \overline{z} + \cdots$$

where $z = \sigma + i_\epsilon b$ and

$$i_\epsilon = \begin{cases} i & \text{for } \epsilon = -1 \\ e & \text{for } \epsilon = 1 \end{cases}$$

- Complexification: both real actions have the same complexification, which is obtained by taking σ and b (or z and $\overline{z}$) to be independent complex fields.

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}_\epsilon = \mathbb{C}^2 \,, \quad \text{for} \quad \epsilon = \pm 1$$

Analytic continuation between the two real forms:

$$b \to -ieb \; : \quad \sigma + ib \to \sigma + eb$$

Special ϵ-complex geometry

- For general target spaces M: ϵ-complex structure

$$J \in \Gamma(\mathrm{End}TM) , \quad J^2 = \epsilon \mathrm{Id}$$

  s.t. eigendistibutions have equal rank.
- Concepts such as 'Hermitean' and 'Kähler' have para-complex analogous.
- I.p. one can define affine and projective special para-Kähler manifolds $=$ Target space geometries of Euclidean $N = 2$ vector multiplets.
- Lagrangian (including gauge fields and fermions) can be written uniformly using ϵ-complex notation.

Example: *STU* model

Prepotential $F = -\frac{X^1 X^2 X^3}{X^0}$.
Special (para-)Kähler manifolds

$$\left(\frac{SL(2,\mathbb{R})}{SO(2)}\right)^3 \subset \left(\frac{SL(2,\mathbb{C})}{GL(1,\mathbb{C})}\right)^3 \supset \left(\frac{SL(2,\mathbb{R})}{SO(1,1)}\right)^3$$

with (para-)Kähler potential

$$K = -\log(S+\overline{S})(T+\overline{T})(U+\overline{U})$$

where

$$S = \epsilon i_\epsilon \frac{X^1}{X^0}\,, \quad T = \epsilon i_\epsilon \frac{X^2}{X^0}\,, \quad U = \epsilon i_\epsilon \frac{X^3}{X^0}\,.$$

Part IV

Instantons

Example

Look for supersymmetric purely scalar solutions of the Euclidean *STU* model. Take $T, U = \text{const}$.

$$S \simeq \int d^4x \sqrt{g} \left(-\frac{1}{2} R - \frac{\partial_m S \partial^m \overline{S}}{(S + \overline{S})^2} + \cdots \right)$$

Imposing 4 Killing spinors:

$$\partial_m \mathrm{Re} S = \pm \partial_m \mathrm{Im} S$$

This implies $T_{mn} = 0$, hence $R_{mn} = 0$, solved by $g_{mn} = \delta_{mn}$. Setting

$$S = e^{-2\phi} + ea$$

the equations of motion reduce to

$$\Delta e^{2\phi} = 0 \; .$$

Comments

- Euclidean supersymmetry $\rightarrow S$ flows along null directions. Equations of motion $\rightarrow S$ defines a harmonic map from S to M. S maps into a completely isotropic, totally geodesic submanifold of M.
- With (positive definit) Kähler target geometry, we do not have null directions, hence $S = \text{const}$. Directly from supersymmetry:

$$\partial_m \text{Re} S = \pm i \partial_m \text{Im} S$$

  for Kähler target space.
- Irrespective of supersymmetry, Derrick's theorem implies that we can only have non-trivial purely scalar solutions for indefinite target space signature.
- Our solution can be viewed as a complex saddle point of the Wick-rotated action.

Lift to a 5d black hole

Take spherically symmetric, 'single centered' solution,

$$e^{2\phi} = e^{2\phi_\infty} + \frac{C}{r^2} , \quad C > 0 .$$

Lift solution to $4+1$ dimensions:

$$ds^2 = -H(r)^{-2/3} dt^2 + H(r)^{1/3} (dr^2 + r^2 d\Omega^2) , \quad H(r) = e^{2(\phi - \phi_\infty)} .$$

Supersymmetric ('small') black hole.

Can be lifted further to a ten-dimensional five-brane.
4d solution = five-brane with all six world-volume direction wrapped.
Suggests interpretation as a stringy instanton.

Interpretation as an instanton

Problems:

1. Expect $S_{\text{inst}} = \frac{|Q_{\text{inst}}|}{e^{2\phi_\infty}}$, but find $S_{\text{inst}} = 0$.
2. The solution is a saddle point of an action which is not bounded from below. How to carry out a saddle point approximation of the functional integral?

Answers are probably well know (though not always well explained in the literature).

Boundary contribution to action

Indefiniteness of the Euclidean action is an essential feature for having (i) non-trivial scalar solutions and (ii) supersymmetric scalar field configurations, (iii) field configurations which lift to 5d black holes.

The instanton action is a boundary term (various ways of derivation). Natural in instanton/soliton correspondence:

$$M^{5d}_{ADM} = S^{4d}_{\mathrm{inst}} = \frac{|Q_{\mathrm{inst}}|}{e^{2\phi_\infty}}$$

where $Q_{\mathrm{inst}} = \pm 2\pi^2 C$.
NB: ADM mass is a boundary term.

Instanton amplitudes

My current understanding (still progressing . . .):

Meaningful saddle point approximation: choose 'integration contour' in complexified field space such that the Gaussian integral is damped.

If we view the solution as a complex saddle point of the positive definit Wick rotated action, the 'integration contour' is shifted be an imaginary constant. Note that we still need to add a boundary term to account for the instanton action.

Concluding remarks

- Special geometry of $N = 2$ vector multiplets = affine/projective special (para)-Kähler geometry.
- Maintaining symplectic covariance crucial.
- Black holes/OSV conjecture: treatment of non-holomorphic corrections: supergravity vs topological string? Derivation of OSV 'measure factor'?
- Euclidean supersymmetry/instantons. Complexification of M and its real forms. Physics: instanton amplitudes, generation of stationary solutions through dimensional lifting.
- Analogous question for hypermultiplets/hypercomplex geometries.