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On holonomy of supermanifolds

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Vector superspace: $V = V_{\bar{0}} \oplus V_{\bar{1}}$ ($\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$)

Homogeneous elements: $x \in V_{\bar{0}} \cup V_{\bar{1}}$

$x \in V_{\bar{0}}$ is called even, $|x| = \bar{0}$;

$x \in V_{\bar{1}} \backslash \{0\}$ is called odd, $|x| = \bar{1}$;

V and W are vector superspaces

$\Rightarrow V \otimes W$ and $\mathrm{Hom}(V, W)$ are vector superspaces:

$(V \otimes W)_{\bar{0}} = (V_{\bar{0}} \otimes W_{\bar{0}}) \oplus (V_{\bar{1}} \otimes W_{\bar{1}}) \qquad (V \otimes W)_{\bar{1}} = (V_{\bar{0}} \otimes W_{\bar{1}}) \oplus (V_{\bar{1}} \otimes W_{\bar{0}})$

$$\begin{aligned}
\mathrm{Hom}(V, W)_{\bar{0}} &= \mathrm{Hom}(V_{\bar{0}}, W_{\bar{0}}) \oplus \mathrm{Hom}(V_{\bar{1}}, W_{\bar{1}}) \\
&= \{f \in \mathrm{Hom}(V, W) \big| \quad |f(x)| = |x|\} \qquad \text{(morphisms)} \\
\mathrm{Hom}(V, W)_{\bar{1}} &= \mathrm{Hom}(V_{\bar{0}}, W_{\bar{1}}) \oplus \mathrm{Hom}(V_{\bar{1}}, W_{\bar{0}}) \\
&= \{f \in \mathrm{Hom}(V, W) \big| \quad |f(x)| = |x| + \bar{1},\ x \neq 0\}
\end{aligned}$$

Superalgebra: $A = A_{\bar{0}} \oplus A_{\bar{1}}$, $\cdot : A \otimes A \to A$, $|xy| = |x| + |y|$

A is called *commutative* if $xy = (-1)^{|x||y|} yx$

Example. The Grassmann superalgebra

$\Lambda(n) = \oplus_{i=0}^{n} \Lambda^i \mathbb{R}^n = \Lambda^{even} \oplus \Lambda^{odd}$ is commutative

Lie superalgebra: $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, $[\cdot, \cdot] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$, $|[x, y]| = |x| + |y|$

1) $[x, y] = (-1)^{|x||y|}[y, x]$

2) $[[x, y], z] + (-1)^{|x|(|y|+|z|)}[[y, z], x] + (-1)^{|z|(|x|+|y|)}[[z, x], y] = 0$

$\Rightarrow$ $\mathfrak{g}_{\bar{0}}$ is a Lie algebra and $\mathfrak{g}_{\bar{1}}$ is a $\mathfrak{g}_{\bar{0}}$-module

Example. $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ $\qquad \mathfrak{gl}(n|m, \mathbb{K}) = \{ \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right) \}$

$\mathfrak{gl}(n|m, \mathbb{K})_{\bar{0}} = \{ \left(\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix} \right) \} \simeq \mathfrak{gl}(n, \mathbb{K}) \oplus \mathfrak{gl}(m, \mathbb{K})$

$\mathfrak{gl}(n|m)_{\bar{1}} = \{ \left(\begin{smallmatrix} 0 & B \\ C & 0 \end{smallmatrix} \right) \} \simeq (\mathbb{K}^n \otimes (\mathbb{K}^m)^*) \oplus ((\mathbb{K}^n)^* \otimes \mathbb{K}^m)$

$[X, Y] = XY - (-1)^{|X||Y|} YX$

Supermanifold: $\mathcal{M}^{n|m} = (M, \mathcal{O}_\mathcal{M})$ $\quad M$ is a smooth n-dim. manifold, $\mathcal{O}_\mathcal{M}$ is a sheaf of superalgebras over $\mathbb{R}$ such that locally

$$\mathcal{O}_\mathcal{M}(U) \simeq \mathcal{O}_M(U) \otimes \Lambda(m)$$

(x^i) $(i = 1, ..., n)$ coordinates on M, (ξ^α) $(\alpha = 1, ..., m)$ a basis of $\mathbb{R}^m$

$\Rightarrow (x^i, \xi^\alpha) = (x^a)$ are called coordinates on $\mathcal{M}$

(put $x^{n+\alpha} = \xi^\alpha$ and assume $a = 1, ..., n+m$)

$f \in \mathcal{O}_M(U) \;\Rightarrow$

$$f = \tilde{f} + \sum_{r=1}^{m} \sum_{\alpha_1 < \cdots < \alpha_r} f_{\alpha_1 \ldots \alpha_r} \xi^{\alpha_1} \cdots \xi^{\alpha_r}, \quad \tilde{f}, f_{\alpha_1 \ldots \alpha_r} \in \mathcal{O}_M(U)$$

$$x \in U \quad \Rightarrow \quad f(x) := \tilde{f}(x)$$

$\Rightarrow f$ is not determined by its values at all points of U!!!

The tangent sheaf: $\mathcal{T}_\mathcal{M} = (\mathcal{T}_\mathcal{M})_{\bar{0}} \oplus (\mathcal{T}_\mathcal{M})_{\bar{1}}$,

$$(\mathcal{T}_\mathcal{M})_{\bar{i}}(U) = \left\{ X : \mathcal{O}_\mathcal{M}(U) \to \mathcal{O}_\mathcal{M}(U) \;\middle|\; \begin{array}{c} |X| = \bar{i}, \; X \text{ is } \mathbb{R}\text{-linear} \\ X(fg) = X(f)g + (-1)^{|f||g|} f X(g) \end{array} \right\}$$

The vector fields $\partial_i = \partial_{x^i}$, $\partial_\alpha = \partial_{\xi^\alpha}$ form a local basis of $\mathcal{T}_\mathcal{M}(U)$

$\Rightarrow \mathcal{T}_\mathcal{M}$ is a locally free sheaf of supermodules over $\mathcal{O}_\mathcal{M}$

Example. $E \to M$ a vector bundle $\Rightarrow \mathcal{O}_\mathcal{M}(U) := \Lambda(\Gamma(U, E))$ defines a supermanifold $\mathcal{M}$.

Let $\mathcal{E}$ be a locally free sheaf of supermodules over $\mathcal{O}_{\mathcal{M}}$ of rank $p|q$.

$x \in M$ consider the fiber at x: $\mathcal{E}_x := \mathcal{E}(U)/(\mathcal{O}_{\mathcal{M}}(U))_x\mathcal{E}(U)$,

where $x \in U$ and $(\mathcal{O}_{\mathcal{M}}(U))_x \subset \mathcal{O}_{\mathcal{M}}(U)$ are functions vanishing at x.

For $X \in \mathcal{E}(U)$ consider the value $X_x \in \mathcal{E}_x$

Example. $\mathcal{E} = \mathcal{T}_{\mathcal{M}} \Rightarrow (\mathcal{T}_{\mathcal{M}})_x = T_x\mathcal{M}$ and $(T_x\mathcal{M})_{\bar{0}} = T_xM$

Consider the vector bundle $E = \cup_{x\in M}\mathcal{E}_x \to M$.

We get the projection $\sim$: $\mathcal{E}(U) \to \Gamma(U, E), \quad X \mapsto \tilde{X}, \quad \tilde{X}_x = X_x$

Let (e_A) $A = 1, ..., p + q$ be a basis of $\mathcal{E}(U)$

$X \in \mathcal{E}(U) \Rightarrow X = X^Ae_A$ $(X^A \in \mathcal{O}_{\mathcal{M}}(U)) \Rightarrow \tilde{X} = \tilde{X}^A\tilde{e}_A$

Connection on $\mathcal{E}$: $\quad \nabla : \mathcal{T}_{\mathcal{M}} \otimes_{\mathbb{R}} \mathcal{E} \to \mathcal{E} \qquad |\nabla_XY| = |X| + |Y|$,

$$\nabla_{fY}X = f\nabla_YX \quad \text{and} \quad \nabla_YfX = (Yf)X + (-1)^{|Y||f|}f\nabla_YX$$

Locally: $\nabla_{\partial_a}e_B = \Gamma^A_{aB}e_A, \quad \Gamma^A_{aB} \in \mathcal{O}_{\mathcal{M}}(U)$

$\tilde{\nabla} = (\nabla|_{\Gamma(TM)\otimes\Gamma(E)})^{\sim} : \Gamma(TM) \otimes \Gamma(E) \to \Gamma(E)$ is a connection on E

$\tilde{\Gamma}^A_{iB}$ are Cristoffel symbols of $\tilde{\nabla}$

$\gamma : [a, b] \subset \mathbb{R} \to M \quad \tau_\gamma : E_{\gamma(a)} \to E_{\gamma(b)}$ the parallel displacement along γ.

$\tau_\gamma : \mathcal{E}_{\gamma(a)} \to \mathcal{E}_{\gamma(b)}$ is an isomorphism of vector superspaces.

Problem: Define holonomy of ∇ (it must give information about all parallel sections of $\mathcal{E}$!)

Parallel sections

$X \in \mathcal{E}(M)$ is called parallel if $\nabla X = 0$

$\nabla X = 0 \Rightarrow \tilde{\nabla}\tilde{X} = 0 \quad (\not\Leftarrow !!!)$

Locally:

$$\nabla X = 0 \Leftrightarrow \begin{cases} \partial_i X^A + X^B \Gamma^A_{iB} = 0, \\ \partial_\gamma X^A + (-1)^{|X^B|} X^B \Gamma^A_{\gamma B} = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} (\partial_{\gamma_r} \ldots \partial_{\gamma_1} (\partial_i X^A + X^B \Gamma^A_{iB}))^\sim = 0, & (*) \\ (\partial_{\gamma_r} \ldots \partial_{\gamma_1} (\partial_\gamma X^A + (-1)^{|X^B|} X^B \Gamma^A_{\gamma B}))^\sim = 0 & (**) \end{cases} \qquad r = 0, \ldots, m$$

$$\tilde{\nabla}\tilde{X} = 0 \Leftrightarrow \partial_i \tilde{X}^A + \tilde{X}^B \tilde{\Gamma}^A_{iB} = 0$$

Prop. A parallel section $X \in \mathcal{E}(M)$ is uniquely defined by its value at any point $x \in M$.

Proof. $\nabla X = 0 \Rightarrow \tilde{\nabla}\tilde{X} = 0$; $\tilde{X}_x = X_x$ uniquely determine $\tilde{X}$, i.e. we know the functions $\tilde{X}^A$.

Further, use $(**)$: $X^A_\gamma = -\tilde{X}^B \tilde{\Gamma}^A_{\gamma B}$,

$X^A_{\gamma\gamma_1} = -\tilde{X}^B \Gamma^A_{\gamma B \gamma_1} + X^B_{\gamma_1} \tilde{\Gamma}^A_{\gamma B} \ldots \Rightarrow$ we know the functions X^A. $\square$

Def. (holonomy algebra) $\mathfrak{hol}(\nabla)_x :=$

$$\left\langle \tau_\gamma^{-1} \circ \bar{\nabla}^r_{Y_r,\dots,Y_1} R_y(Y,Z) \circ \tau_\gamma \;\middle|\; \begin{array}{l} r \geq 0,\ Y, Z, Y_i \in T_y\mathcal{M} \\ \bar{\nabla}\text{: connect on } \mathcal{T}_\mathcal{M}|_U \end{array} \right\rangle \subset \mathfrak{gl}(\mathcal{E}_x) \simeq \mathfrak{gl}(p|q,\mathbb{R})$$

Note: $\mathfrak{hol}(\tilde{\nabla})_x \subset (\mathfrak{hol}(\nabla)_x)_{\bar{0}} \qquad (\neq !)$

Lie supergroup $\mathcal{G} = (G, \mathcal{O}_\mathcal{G})$ is a group object in the category of supermanifolds; $\mathcal{G}$ is uniquely given by the Harish-Chandra pair $(G, \mathfrak{g})$, where $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is a Lie superalgebra, $\mathfrak{g}_{\bar{0}}$ is the Lie algebra of G.

Denote by $\mathrm{Hol}(\nabla)^0_x$ the connected Lie subgroup of $\mathrm{GL}((\mathcal{E}_x)_{\bar{0}}) \times \mathrm{GL}((\mathcal{E}_x)_{\bar{1}})$ corresponding to $(\mathfrak{hol}(\nabla)_x)_{\bar{0}} \subset \mathfrak{gl}((\mathcal{E}_x)_{\bar{0}}) \oplus \mathfrak{gl}((\mathcal{E}_x)_{\bar{1}}) \subset \mathfrak{gl}(\mathcal{E}_x)$;

$\mathrm{Hol}(\nabla)_x := \mathrm{Hol}(\nabla)^0_x \cdot \mathrm{Hol}(\tilde{\nabla})_x \subset \mathrm{GL}((\mathcal{E}_x)_{\bar{0}}) \times \mathrm{GL}((\mathcal{E}_x)_{\bar{1}})$.

Def. Holonomy group: $\mathcal{H}ol(\nabla)_x := (\mathrm{Hol}(\nabla)_x, \mathfrak{hol}(\nabla)_x)$;

the restricted holonomy group: $\mathcal{H}ol(\nabla)^0_x := (\mathrm{Hol}(\nabla)^0_x, \mathfrak{hol}(\nabla)_x)$.

Def. (infinitesimal holonomy algebra) $\mathfrak{hol}(\nabla)^{inf}_x :=$

$$< \tau_\gamma^{-1} \circ \bar{\nabla}^r_{Y_r,\dots,Y_1} R_x(Y,Z) \circ \tau_\gamma | r \geq 0,\ Y, Z, Y_1, \dots, Y_r \in T_x\mathcal{M} > \subset \mathfrak{hol}(\nabla)_x$$

Theorem. If $\mathcal{M}$, $\mathcal{E}$ and ∇ are analytic, then $\mathfrak{hol}(\nabla)_x = \mathfrak{hol}(\nabla)^{inf}_x$.

Theorem.

$$\{X \in \mathcal{E}(M),\ \ \nabla X = 0\} \longleftrightarrow \left\{ \begin{array}{c} X_x \in \mathcal{E}_x \text{ annihilated by } \mathfrak{hol}(\nabla)_x \\ \text{and preserved by } \operatorname{Hol}(\tilde{\nabla})_x \end{array} \right\}$$

Proof. $\longrightarrow$: $\quad \nabla X = 0 \quad \Rightarrow \quad \bar{\nabla}^r_{Y_r,\dots,Y_1} R(Y,Z)X = 0$

$\nabla X = 0 \quad \Rightarrow \quad \tilde{\nabla}\tilde{X} = 0 \quad \Rightarrow \quad \tilde{X}$ is preserved by $\operatorname{Hol}(\tilde{\nabla})_x$

$\Longrightarrow \quad \bar{\nabla}^r_{Y_r,\dots,Y_1} R_y(Y,Z) \circ \tau_\gamma X_x = 0 \quad \Rightarrow \quad X_x$ is annihilated by $\mathfrak{hol}(\nabla)_x$

$\longleftarrow$:

$\operatorname{Hol}(\tilde{\nabla})_x$ preserves $X_x \in \mathcal{E}_x \Longrightarrow\ \exists X_0 \in \Gamma(E),\ \tilde{\nabla} X_0 = 0,\ (X_0)_x = X_x$

$X_0 = X_0^A \tilde{e}_A,\ X_0^A \in \mathcal{O}_M(U)$

$(**)$ defines $X^A_{\gamma\gamma_1\dots\gamma_r} \in \mathcal{O}_M(U)$ for all $\gamma < \gamma_1 < \dots < \gamma_r$, $0 \le r \le m-1$.

We get $X^A \in \mathcal{O}_\mathcal{M}(U)$, $\qquad$ consider $X = X^A e_A \in \mathcal{E}(U)$.

Claim: $\nabla X = 0$. $\qquad$ To prove (by induction over r):

X^A satisfy $(*)$ and $(**)$ for all $\gamma_1 < \dots < \gamma_r$, $0 \le r \le m$

$$\begin{aligned}(\partial_{\gamma_r}\dots\partial_{\gamma_1}(\partial_i X^A + X^B \Gamma^A_{iB}))^\sim &= (\partial_{\gamma_r}\dots\partial_{\gamma_2}((-1)^{(|A|+|B|)|X^B|} R^A_{B\gamma_1 i} X^B))^\sim \\ &= (\partial_{\gamma_r}\dots\partial_{\gamma_3}((-1)^{(|A|+|B|)|X^B|} \bar{\nabla}_{\gamma_2} R^A_{B\gamma_1 i} X^B))^\sim \\ &= \dots = ((-1)^{(|A|+|B|)|X^B|} \bar{\nabla}^{r-1}_{\gamma_r,\dots,\gamma_2} R^A_{B\gamma_1 i} X^B)^\sim = 0,\end{aligned}$$

this proves $(*)$

Parallel subsheaves

A subsheaf $\mathcal{F} \subset \mathcal{E}$ of $\mathcal{O}_\mathcal{M}$-supermodules is called *a locally direct* if locally there exists a basis of $\mathcal{E}(U)$ some elements of which form a basis of $\mathcal{F}(U)$

A distribution on $\mathcal{M}$ is a locally direct subsheaf of $\mathcal{T}_\mathcal{M}$

$\mathcal{F} \subset \mathcal{E}$ is *parallel* if $\nabla_Y X \in \mathcal{F}(U)$ for all $Y \in \mathcal{T}_\mathcal{M}(U)$ and $X \in \mathcal{F}(U)$

Theorem.

$$\{\text{parallel locally direct subsheaves } \mathcal{F} \subset \mathcal{E} \text{ of rank } p_1|q_1\}$$

$$\longleftrightarrow \quad \{\mathcal{F}_x \subset \mathcal{E}_x \text{ of dimension } p_1|q_1 \text{ preserved by } \mathfrak{hol}(\nabla)_x \text{ and } \operatorname{Hol}(\tilde{\nabla})_x\}$$

Linear connections

∇ a connection on $\mathcal{E} = \mathcal{T}_{\mathcal{M}}$, $\quad E = \cup_{y\in M} T_y\mathcal{M} = T\mathcal{M}$, $\quad E_{\bar 0} = TM$

$\mathfrak{hol}(\nabla) \subset \mathfrak{gl}(n|m,\mathbb{R})$, $\quad \mathrm{Hol}(\tilde\nabla) \subset \mathrm{GL}(n,\mathbb{R}) \times \mathrm{GL}(m,\mathbb{R})$

Theorem.

$$\left\{\begin{matrix}\text{Parallel tensor fields}\\ \text{of type } (p,q) \text{ on } \mathcal{M}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}A_x \in T^{p,q}_x\mathcal{M} \text{ annihilated by } \mathfrak{hol}(\nabla)_x\\ \text{and preserved by } \mathrm{Hol}(\tilde\nabla)_x\end{matrix}\right\}$$

Let g be a bilinear form on a vector superspace V.

g is *even* if $g(V_{\bar 0}, V_{\bar 1}) = g(V_{\bar 1}, V_{\bar 0}) = 0$

g is *odd* if $g(V_{\bar 0}, V_{\bar 0}) = g(V_{\bar 1}, V_{\bar 1}) = 0$

g is *supersymmetric* if $g(x,y) = (-1)^{|x||y|} g(y,x)$

g is *super skew-symmetric* if $g(x,y) = -(-1)^{|x||y|} g(y,x)$

Example. Let g be non-degenerate even and supersymmetric

$\Rightarrow g|_{V_{\bar 0}\times V_{\bar 0}}$ is a usual non-degenerate symmetric bilinear form (of sign. (p,q))

and $g|_{V_{\bar 1}\times V_{\bar 1}}$ is a usual non-degenerate skew-symmetric bilinear form

$\mathfrak{so}(p,q|2k,\mathbb{R})$ is a subalgebra of $\mathfrak{gl}(p+q|2k,\mathbb{R})$ preserving g

$$\mathfrak{so}(p,q|2k,\mathbb{R}) = \left\{ \left(\begin{matrix} A & B_1 & B_2 \\ -B_2^t & C_1 & C_2 \\ B_1^t & C_3 & -C_1^t \end{matrix} \right) \middle| A \in \mathfrak{so}(p,q),\ C_2^t = C_2,\ C_3^t = C_3 \right\}$$

$\mathfrak{so}(p,q|2k,\mathbb{R})_{\bar 0} \simeq \mathfrak{so}(p,q) \oplus \mathfrak{sp}(2k,\mathbb{R})$, $\quad \mathfrak{so}(p,q|2k,\mathbb{R})_{\bar 1} \simeq \mathbb{R}^{p+q} \otimes \mathbb{R}^{2k}$

Examples of parallel structures on $(\mathcal{M}, \nabla)$ and the corresponding holonomy

parallel structure on $\mathcal{M}$	$\mathfrak{hol}(\nabla)$ is contained in	$\mathrm{Hol}(\tilde{\nabla})$ is contained in	restriction
complex structure	$\mathfrak{gl}(k\|l, \mathbb{C})$	$\mathrm{GL}(k, \mathbb{C}) \times \mathrm{GL}(l, \mathbb{C})$	$n = 2k, l = 2m$
odd complex structure, i.e. odd automorphism J of $\mathcal{T}_{\mathcal{M}}$ with $J^2 = -\mathrm{id}$	$\mathfrak{q}(n, \mathbb{R})$ (queer Lie superalgebra)	$\left\{ \left(\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix} \right) \middle\| A \in \mathrm{GL}(n, \mathbb{R}) \right\}$	$m = n$
Riemannian supermetric, i.e. even non-degenerate supersymmetric metric	$\mathfrak{osp}(p_0, q_0\|2k)$	$\mathrm{O}(p_0, q_0) \times \mathrm{Sp}(2k, \mathbb{R})$	$n = p_0 + q_0, m = 2k$
even non-degenerate super skew-symmetric metric	$\mathfrak{osp}^{\mathrm{sk}}(2k\|p, q)$	$\mathrm{Sp}(2k, \mathbb{R}) \times \mathrm{O}(p, q)$	$n = 2k, m = p + q$
odd non-degenerate supersymmetric metric	$\mathfrak{pe}(n, \mathbb{R})$ (periplectic Lie superalgebra)	$\left\{ \left(\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix} \right) \middle\| A \in \mathrm{GL}(n, \mathbb{R}) \right\}$	$m = n$
odd non-degenerate super skew-symmetric metric	$\mathfrak{pe}^{sk}(n, \mathbb{R})$	$\left\{ \left(\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix} \right) \middle\| A \in \mathrm{GL}(n, \mathbb{R}) \right\}$	$m = n$

Riemannian supermanifolds

On $(\mathcal{M}, g)$ exists a unique Levi-Civita connection ∇

$\mathfrak{hol}(\mathcal{M}, g) \subset \mathfrak{osp}(p_0, q_0|2k)$ and $\mathrm{Hol}(\tilde{\nabla}) \subset \mathrm{O}(p_0, q_0) \times \mathrm{Sp}(2k, \mathbb{R})$

Special geometries of Riemannian supermanifolds and the corresponding holonomies

type of $(\mathcal{M}, g)$	$\mathfrak{hol}(\mathcal{M}, g)$ is contained in	$\mathrm{Hol}(\tilde{\nabla})$ is contained in	restriction
Kählerian	$\mathfrak{u}(p_0, q_0\|p_1, q_1)$	$\mathrm{U}(p_0, q_0) \times \mathrm{U}(p_1, q_1)$	$n = 2p_0 + 2q_0$, $m = 2p_1 + 2q_1$
special Kählerian (by def.)	$\mathfrak{su}(p_0, q_0\|p_1, q_1)$	$\mathrm{U}(1)(\mathrm{SU}(p_0, q_0) \times \mathrm{SU}(p_1, q_1))$	$n = 2p_0 + 2q_0$, $m = 2p_1 + 2q_1$
hyper-Kählerian	$\mathfrak{hosp}(p_0, q_0\|p_1, q_1)$	$\mathrm{Sp}(p_0, q_0) \times \mathrm{Sp}(p_1, q_1)$	$n = 4p_0 + 4q_0$, $m = 4p_1 + 4q_1$
quaternionic-Kählerian	$\mathfrak{sp}(1) \oplus \mathfrak{hosp}(p_0, q_0\|p_1, q_1)$	$\mathrm{Sp}(1)(\mathrm{Sp}(p_0, q_0) \times \mathrm{Sp}(p_1, q_1))$	$n = 4p_0 + 4q_0 \geq 8$, $m = 4p_1 + 4q_1$

$\mathrm{Ric}(Y, Z) := \mathrm{str}\big(X \mapsto (-1)^{|X||Z|} R(Y, X)Z\big), \qquad \mathrm{str}\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) = \mathrm{tr}\, A - \mathrm{tr}\, D$

Prop. Let $(\mathcal{M}, g)$ be a Kählerian supermanifold, then $\mathrm{Ric} = 0$ if and only if $\mathfrak{hol}(\mathcal{M}, g) \subset \mathfrak{su}(p_0, q_0|p_1, q_1)$. In particular, if $(\mathcal{M}, g)$ is special Kählerian, then $\mathrm{Ric} = 0$; if M is simply connected, $(\mathcal{M}, g)$ is Kählerian and $\mathrm{Ric} = 0$, then $(\mathcal{M}, g)$ is special Kählerian.

A generalization of the Wu theorem

the product $\mathcal{M}\times\mathcal{N}=(M\times N,\mathcal{O}_{\mathcal{M}\times\mathcal{N}})$:

Let $(U,x^1,...,x^n,\xi^1,...,\xi^m)$ and $(V,y^1,...,y^p,\eta^1,...,\eta^q)$ be coordinate systems on $\mathcal{M}$ and $\mathcal{N}$

by definition, $\mathcal{O}_{\mathcal{M}\times\mathcal{N}}(U\times V):=\mathcal{O}_{M\times N}(U\times V)\otimes\Lambda_{\xi^1,...,\xi^m,\eta^1,...,\eta^q}$

a supersubalgebra $\mathfrak{g}\subset\mathfrak{osp}(p_0,q_0|2k)$ is *weakly-irreducible* if it does not preserve any non-degenerate vector supersubspace of $\mathbb{R}^{p_0+q_0}\oplus\Pi(\mathbb{R}^{2k})$.

Theorem. Let $(\mathcal{M},g)$ be a Riemannian supermanifold such that the pseudo-Riemannian manifold $(M,\tilde{g})$ is simply connected and geodesically complete. Then there exist Riemannian supermanifolds

$(\mathcal{M}_0,g_0),(\mathcal{M}_1,g_1),...,(\mathcal{M}_r,g_r)$ such that

$$(\mathcal{M},g)=(\mathcal{M}_0\times\mathcal{M}_1\times\cdots\times\mathcal{M}_r,g_0+g_1+\cdots+g_r),\quad (1)$$

the supermanifold $(\mathcal{M}_0,g_0)$ is flat and the holonomy algebras of the supermanifolds $(\mathcal{M}_1,g_1)$,...,$(\mathcal{M}_r,g_r)$ are weakly-irreducible. In particular,

$$\mathfrak{hol}(\mathcal{M},g)=\mathfrak{hol}(\mathcal{M}_1,g_1)\oplus\cdots\oplus\mathfrak{hol}(\mathcal{M}_r,g_r).$$

For general $(\mathcal{M},g)$ decomposition (1) holds locally.

Proof. *local version:* $x \in M$, if $\mathfrak{hol}(\mathcal{M}, g)_x$ is not weakly-irreducible, then $\mathfrak{hol}(\mathcal{M}, g)_x$ preserves $F_1, F_2 \subset T_x\mathcal{M}, \quad F_1 \oplus F_2 = T_x\mathcal{M}$

$\Rightarrow \exists$ parallel distributions $\mathcal{F}_1$ and $\mathcal{F}_2$ over $\mathcal{M}$

$\Rightarrow \mathcal{F}_1$ and $\mathcal{F}_2$ are involutive $\Rightarrow \exists$ maximal integral submanifolds $\mathcal{M}_1$ and $\mathcal{M}_2$ of $\mathcal{M}$ passing through the point x

$\Rightarrow \exists$ local coordinates $x^1, ..., x^n, \xi^1, ..., \xi^m$ (resp., $y^1, ..., y^n, \eta^1, ..., \eta^m$) on $\mathcal{M}$ such that $x^1, ..., x^{n_1}, \xi^1, ..., \xi^{m_1}$ (resp., $y^1, ..., y^{n-n_1}, \eta^1, ..., \eta^{m-m_1}$) are coordinates on $\mathcal{M}_1$ (resp., on $\mathcal{M}_2$).

$\Rightarrow x^1, ..., x^{n_1}, y^1, ..., y^{n-n_1}, \xi^1, ..., \xi^{m_1}, \eta^1, ..., \eta^{m-m_1}$ are coordinates on $\mathcal{M}$ and $\mathcal{M}$ is locally isomorphic to a domain in the product $\mathcal{M}_1 \times \mathcal{M}_2$.

g_1 and g_2 do not depend on the coordinates $y^1, ..., y^{n-n_1}, \eta^1, ..., \eta^{m-m_1}$ and $x^1, ..., x^{n_1}, \xi^1, ..., \xi^{m_1}$, respectively.

$(\mathcal{M}_1, g_1)$ and $(\mathcal{M}_2, g_2)$ are Riemannian supermanifolds and $g = g_1 + g_2$.

global version:

$(F_1)_{\bar{0}}, (F_2)_{\bar{0}} \subset T_xM$ are non-degenerate and preserved by $\mathrm{Hol}(M, \tilde{g})_x$

the Wu theorem $\Rightarrow M \simeq M_1 \times M_2$

the underlying manifolds of the supermanifolds $\mathcal{M}_1$ and $\mathcal{M}_2$ are M_1 and M_2, respectively

local version $\Rightarrow \mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$ and $g = g_1 + g_2$

Berger superalgebras

Problem: Classify possible irreducible holonomy algebras of torsion-free linear connections

V a vector superspace, $\mathfrak{g} \subset \mathfrak{gl}(V)$ a supersubalgebra

The space of algebraic curvature tensors of type $\mathfrak{g}$:

$$\mathcal{R}(\mathfrak{g}) = \left\{ R \in V^* \wedge V^* \otimes \mathfrak{g} \,\middle|\, \begin{array}{c} R(X,Y)Z + (-1)^{|X|(|Y|+|Z|)} R(Y,Z)X \\ +(-1)^{|Z|(|X|+|Y|)} R(Z,X)Y = 0 \\ \text{for all homogeneous } X,Y,Z \in V \end{array} \right\}$$

$\mathfrak{g} \subset \mathfrak{gl}(V)$ is *a Berger superalgebra* if

$$\mathrm{span}\{R(X,Y) | R \in \mathcal{R}(\mathfrak{g}),\ X,Y \in V\} = \mathfrak{g}$$

Prop. Let $\mathcal{M}$ be a supermanifold of dimension $n|m$ with a linear torsion-free connection ∇. Then its holonomy algebra $\mathfrak{hol}(\nabla) \subset \mathfrak{gl}(n|m,\mathbb{R})$ is a Berger superalgebra.

Examples of Berger superalgebras

$\mathfrak{g}_0 \subset \mathfrak{gl}(V), \quad V := \mathfrak{g}_{-1}$ The k-prolongation:

$$\mathfrak{g}_k := \{\varphi \in \mathrm{Hom}(\mathfrak{g}_{-1}, \mathfrak{g}_{k-1}) | \varphi(x)y = (-1)^{|x||y|}\varphi(y)x\} \quad (k \geq 1)$$

$$0 \longrightarrow \mathfrak{g}_2 \longrightarrow \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_1 \longrightarrow \mathcal{R}(\mathfrak{g}_0) \longrightarrow H^{2,2}_{\mathfrak{g}_0} \longrightarrow 0$$

Computation of $H^{2,2}_{\mathfrak{g}_0}$: Leites, Serganova, Poletaeva...

Prop. The following are Berger superalgebras:

1) $\mathfrak{gl}(n|m)$, $\mathfrak{sl}(n|m)$, $\mathfrak{osp}^{sk}(n|2m)$ and $\mathfrak{spe}^{sk}(k)$ $(k \geq 3)$

2) $\mathfrak{c}(\mathfrak{sl}(n-p|q) \oplus \mathfrak{sl}(p|m-q))$ and $\mathfrak{sl}(n-p|q) \oplus \mathfrak{sl}(p|m-q)$

if $n \neq m, n-p+q \geq 2, m-q+p \geq 2$,

$\mathfrak{sl}(n-p|q) \oplus \mathfrak{sl}(p|n-q)$ if $n \geq 3, n-p+q \geq 2, n-q+p \geq 2$,

$\mathfrak{cosp}(n|2k)$, $\mathfrak{osp}(n|2k)$, $\mathfrak{ps}(\mathfrak{q}(p) \oplus \mathfrak{q}(n-p))$ and $\mathfrak{p}(\mathfrak{sq}(p) \oplus \mathfrak{sq}(n-p))$;

3) $\mathfrak{gl}(l|k)$ and $\mathfrak{sl}(l|k)$ acting on $\Lambda^2(\mathbb{R}^l \oplus \Pi(\mathbb{R}^k))$;

4) $\mathfrak{sl}(p|n-p)$ acting on both

$\Pi(S^2(\mathbb{R}^p \oplus \Pi(\mathbb{R}^{n-p})))$ and $\Pi(\Lambda^2(\mathbb{R}^p \oplus \Pi(\mathbb{R}^{n-p})))$;

5) $\mathfrak{spe}(n), \mathfrak{pe}(n), \mathfrak{cspe}(n), \mathfrak{cpe}(n)$;

Prop. Let $\mathfrak{g}_0$ be a simple complex Lie superalgebra, $\mathfrak{g}_{-1} = \Pi(\mathfrak{g}_0)$, then $\mathfrak{g}_1 \simeq \Pi(\mathbb{C})$, $\mathfrak{g}_2 = 0$ and $\mathfrak{g}_0 \subset \mathfrak{gl}(\Pi(\mathfrak{g}_0))$ is a Berger superalgebra.