Contact manifolds and $SU(n)$-structures

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$SU(2)$-structures in 5-dimensions

Definition

An $SU(2)$-structure $(\eta, \omega_1, \omega_2, \omega_3)$ on N^5 is given by a 1-form η and by three 2-forms ω_i such that

$$\omega_i \wedge \omega_j = \delta_{ij} v, \; v \wedge \eta \neq 0,$$
$$i_X \omega_3 = i_Y \omega_1 \Rightarrow \omega_2(X, Y) \geq 0,$$

where i_X denotes the contraction by X.

Remark

The pair (η, ω_3) defines a $U(2)$-structure or an almost contact metric structure on N^5, i.e. (η, ξ, φ, g) such that

$$\eta(\xi) = 1, \quad \varphi^2 = -\mathrm{Id} + \xi \otimes \eta,$$
$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y).$$

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Sasaki-Einstein structures

Example

Sasaki-Einstein structure

$$d\eta = -2\omega_3,\ d\omega_1 = 3\eta\wedge\omega_2,\ d\omega_2 = -3\eta\wedge\omega_1.$$

On $S^2\times S^3$ there exist an infinite family of explicit Sasaki-Einstein metrics [Gauntlett, Martelli, Sparks, Waldram, ...].

Definition (Boyer, Galicki)

(N^{2n+1}, g, η) is Sasaki-Einstein if the conic metric $\tilde g = dr^2 + r^2 g$ on the symplectic cone $N^5\times\mathbb{R}^+$ is Kähler and Ricci- flat (CY).

• $N^{2n+1}\times\mathbb{R}^+$ has an integrable $SU(n+1)$-structure, i.e. an Hermitian structure $(J,\tilde g)$, with $F = d(r^2\eta)$, and a $(n+1,0)$-form $\Psi = \Psi_+ + i\Psi_-$ of lenght 1 such that $dF = d\Psi = 0 \Rightarrow \tilde g$ has holonomy in $SU(n+1)$.

• N^{2n+1} has a real Killing spinor, i.e. the restriction of a parallel spinor on the Riemannian cone [Friedrich, Kath].

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Remark

An $SU(2)$-structure P on N^5 induces a spin structure on N^5 and P extends to $P_{Spin(5)} = P \times_{SU(2)} Spin(5)$.

The spinor bundle is $P \times_{SU(2)} \Sigma$, where $\Sigma \cong \mathbb{C}^4$ and $Spin(5)$ acts transitively on the sphere in Σ with stabilizer $SU(2)$ in a fixed unit spinor $u_0 \in \Sigma$.

Then the $SU(2)$-structures are in one-to-one correspondence with the pairs $(P_{Spin(5)}, \psi)$, with ψ a unit spinor such that $\psi = [u, u_0]$ for any local section u of P, i.e. $\psi \in P \times_{Spin(5)} (Spin(5)u_0)$.

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Hypo structures

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An $SU(2)$-structure on N^5 is hypo if

$$d\omega_3 = 0, \; d(\eta \wedge \omega_1) = 0, \; d(\eta \wedge \omega_2) = 0.$$

Proposition (Conti, Salamon)

An $SU(2)$-structure P on N^5 is hypo if and only if the spinor ψ (defined by P) is generalized Killing (in the sense of Bär, Gauduchon, Moroianu) , i.e.

$$\nabla_X \psi = \frac{1}{2} O(X) \cdot \psi,$$

where O is a section of $\mathrm{Sym}(TN^5)$ and $\cdot$ is the Clifford multiplication.

If N^5 is simply connected and Sasaki-Einstein, then $O = \pm \mathrm{Id}$ [Friedrich, Kath].

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• Any oriented hypersurface N^5 of (M^6, F, Ψ) with an integrable $SU(3)$-structure (F, Ψ) has in a natural way a hypo structure.

The generalized Killing spinor ψ on N^5 is the restriction of the parallel spinor on M^6 and O is just given by the Weingarten operator. If ψ is the restriction of a parallel spinor over the Riemannian cone then O is a constant multiple of the identity.

• Nilmanifolds cannot admit Sasaki-Einstein structures but they can admit hypo structures.

Theorem (Conti, Salamon)

The nilpotent Lie algebras admitting a hypo structure are

$$\begin{array}{ll} (0,0,12,13,14), & (0,0,0,12,13+24), \\ (0,0,0,12,13), & (0,0,0,0,12+34), \\ (0,0,0,0,12), & (0,0,0,0,0). \end{array}$$

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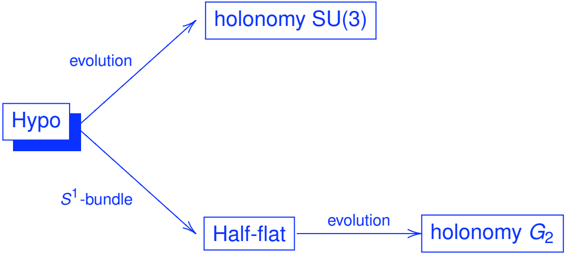

Hypo → $SU(n)$-structures in $(2n+1)$-dimensions → symplectic cone

Hypo evolution equations

Theorem (Conti, Salamon)

A real analytic hypo structure (η, ω_i) on N^5 determines an integrable $SU(3)$-structure on $N^5 \times I$, with I some open interval, if (η, ω_i) belongs to a one-parameter family of hypo structures $(\eta(t), \omega_i(t))$ which satisfy the evolution equations

$$\begin{cases} \partial_t \omega_3(t) = -\hat{d}\eta(t), \\ \partial_t(\omega_2(t) \wedge \eta(t)) = \hat{d}\omega_1(t), \\ \partial_t(\omega_1(t) \wedge \eta(t)) = -\hat{d}\omega_2(t). \end{cases}$$

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η-Einstein structures

Definition

An almost contact metric manifold $(N^{2n+1}, \eta, \xi, \varphi, g)$ is η-Einstein if there exist $a, b \in \mathcal{C}^\infty(N^{2n+1})$ such that

$$\mathrm{Ric}_g(X, Y) = a\, g(X, Y) + b\, \eta(X)\eta(Y),$$

where $\mathrm{scal}_g = a(2n+1) + b$ and $\mathrm{Ric}_g(\xi, \xi) = a + b$.

If $b = 0$, a Sasaki η-Einstein is Sasaki- Einstein.

Theorem (Conti, Salamon)

A hypo structure on N^5 is η-Einstein $\Leftrightarrow$ it is Sasakian.

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with a and b constants [Friedrich, Kim].

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In general, for a hypo structure the 1-form η is not a contact form.

A hypo structure is contact if and only if $d\eta = -2\omega_3$.

Problem

Find examples of manifolds N^5 with a hypo-contact structure.

Examples

- Sasaki η-Einstein manifolds.

An example is given by the nilmanifold associated to

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Classification in the hypo-contact case

Theorem (De Andres, Fernandez, –, Ugarte)

A 5-dimensional solvable Lie algebra $\mathfrak{g}$ has a hypo-contact structure $\Leftrightarrow$ $\mathfrak{g}$ is isomorphic to one of the following:

$$\mathfrak{g}_1 : \quad [e_1, e_4] = [e_2, e_3] = e_5 \ (\textit{nilpotent and } \eta\textit{-Einstein});$$

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New metrics with holonomy $SU(3)$

Studying the Conti-Salamon evolution equations for the left-invariant hypo-contact structures on the simply-connected solvable Lie groups G_i $(1 \leq i \leq 5)$ with Lie algebra $\mathfrak{g}_i$:

Theorem (De Andres, Fernandez, –, Ugarte)

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New metrics with holonomy $SU(3)$

Studying the Conti-Salamon evolution equations for the left-invariant hypo-contact structures on the simply-connected solvable Lie groups G_i $(1 \leq i \leq 5)$ with Lie algebra $\mathfrak{g}_i$:

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Sasakian 5-dimensional Lie algebras

Theorem (Andrada, –, Vezzoni)

Let $\mathfrak{g}$ be a 5-dimensional Sasakian Lie algebra. Then

1. *if $\mathfrak{z}(\mathfrak{g}) \neq \{0\}$, $\mathfrak{g}$ is solvable with $\dim \mathfrak{z}(\mathfrak{g}) = 1$ and the quotient $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ carries an induced Kähler structure;*
2. *if $\mathfrak{z}(\mathfrak{g}) = \{0\}$, $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{aff}(\mathbb{R})$, or $\mathfrak{su}(2) \times \mathfrak{aff}(\mathbb{R})$, or $\mathfrak{g}_3 \cong \mathbb{R}^2 \ltimes \mathfrak{h}_3$, where $\mathfrak{aff}(\mathbb{R})$ is the Lie algebra of the Lie group of affine motions of $\mathbb{R}$.*

• $\mathfrak{g}$ is either solvable or a direct sum.

• A 5-dimensional Sasakian solvmanifold is either a compact quotient of H_5 or of $\mathbb{R} \ltimes (H_3 \times \mathbb{R})$ with structure equations

$$(0, -13, 12, 0, 14 + 23).$$

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• A 5-dimensional Sasakian η-Einstein Lie algebra is isomorphic either to $\mathfrak{g}_1 \cong \mathfrak{h}_5, \mathfrak{g}_3$ or to $\mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{aff}(\mathbb{R})$.

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Definition (Chiossi, Salamon)

An $SU(3)$-structure $(F, \Psi = \Psi_+ + i\Psi_-)$ on M^6 is half-flat if

$$d(F \wedge F) = 0, \; d(\Psi_+) = 0.$$

Theorem (Hitchin)

If the half-flat structure (F, Ψ) belongs to a one-parameter family $(F(t), \Psi(t)$ of half-flat structures, with t in a open interval I, which satisfy the evolution equations

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From hypo to half-flat

Proposition (De Andres, Fernandez, –, Ugarte)

$(N^5, \eta, \omega_1, \omega_2, \omega_3)$ hypo

Ω: integer closed 2-form which annihilates both ω_3 and $\cos\theta\,\omega_1 + \sin\theta\,\omega_2$, for some θ

$\Rightarrow \exists$ a principal S^1-bundle $\pi : M^6 \longrightarrow N^5$ with connection form ρ such that Ω is the curvature of ρ and the $SU(3)$-structure

$$F^\theta = \pi^*(\cos\theta\,\omega_1 + \sin\theta\,\omega_2) + \pi^*(\eta) \wedge \rho,$$

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Theorem (De Andres, Fernandez, –, Ugarte)

$(M^6, F, \Psi = \Psi_+ + i\Psi_-)$ *half-flat*

$\iota : N^5 \hookrightarrow M^6$ *oriented hypersurface with unit normal vector field* $\mathbb{U}$.

- *If* $g(\nabla_{\mathbb{U}}\mathbb{U}, X) = 0$ *and* $\mathcal{L}_{\mathbb{U}}\Psi_+ = 0$, *for any* X *on* N^5, *then the forms*

$$\eta = -i_{\mathbb{U}}F, \quad \omega_1 = -i_{\mathbb{U}}\Psi_-, \quad \omega_2 = \iota^* F, \quad \omega_3 = -i_{\mathbb{U}}\Psi_+,$$

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New metrics with holonomy G_2

We consider S^1-bundles $K_i \to G_i$ with a half-flat structure induced by a hypo-contact structure on G_i.

Solving the Hitchin evolution equations

Theorem (De Andres, Fernandez, –, Ugarte)

The half-flat structure on K_i $(i = 1, 4, 5)$ gives rise to a Riemanian metric with holonomy G_2 on $K_i \times I$, for some open interval I.

- K_1 is nilpotent and the metric with holonomy G_2 on $K_1 \times I$ is known [Chiossi, –].
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$SU(n)$-structures in $(2n+1)$-dimensions

Definition

An $SU(n)$-structure (η, ϕ, Ω) on N^{2n+1} is determined by the forms

$$\eta = e^{2n+1}, \quad \phi = e^1 \wedge e^2 + \ldots + e^{2n-1} \wedge e^{2n},$$
$$\Omega = (e^1 + ie^2) \wedge \ldots \wedge (e^{2n-1} + ie^{2n}).$$

As for the case of $SU(2)$-structures in dimensions 5 we have that an $SU(n)$-structure P_{SU} on N^{2n+1} induces a spin structure P_{Spin} and if we fix a unit element $u_0 \in \Sigma = (\mathbb{C}^2)^{\otimes 2n}$ we have that

$$P_{SU} = \{u \in P_{Spin} \,|\, [u, u_0] = \psi\}.$$

The pair (η, ϕ) defines a $U(n)$-structure or an almost contact metric structure on N^{2n+1}.

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Generalized Killing spinors

Example

$N^{2n+1} \hookrightarrow M^{2n+2}$ (with holonomy in $SU(n+1)$).
Then the restriction of the parallel spinor defines an $SU(n)$-structure (η, ϕ, Ω) where the forms ϕ and $\Omega \wedge \eta$ are the pull-back of the Kähler form and the complex volume form on the CY manifold M^{2n+2}.

Proposition (Conti, –)

Let N^{2n+1} be a real analytic manifold with a real analytc $SU(n)$-structure P_{SU} defined by (η, ϕ, Ω). The following are equivalent:

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The assumption of real analycity is certainly necessary to prove that (1) or (2) implies (3), but the fact that (1) implies (2) does not require this hypothesis.

(2) $\Rightarrow$ (3) can be described in terms of evolution equations in the sense of Hitchin. Indeed, suppose that there is a family $(\eta(t), \phi(t), \Omega(t))$ of $SU(n)$-structures on N^{2n+1}, with t in some interval I, then the forms

$$\eta(t) \wedge dt + \phi(t), \quad (\eta(t) + idt) \wedge \Omega(t)$$

define a CY structure on $N^{2n+1} \times I$ if and only if (2) holds for $t = 0$ and the evolution equations

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An $SU(n)$-structure (η, ϕ, Ω) on N^{2n+1} is contact if $d\eta = -2\phi$.

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If N^{2n+1} is Sasaki-Einstein, we know that the symplectic cone is CY with the cone metric $r^2 g + dr^2$ and the Kähler form equal to the conical symplectic form.

Problem

If one thinks the form ϕ as the pullback to $N^{2n+1} \cong N^{2n+1} \times \{1\}$ of the conical symplectic form, which types of contact $SU(n)$-structures give rise to a CY symplectic cone but not necessarily with respect to the cone metric?

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Examples

• 5-dimensional hypo-contact solvable Lie groups [De Andres, Fernandez, –, Ugarte].

• The $(2n+1)$-dimensional real Heisenberg Lie group

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- 5-dimensional hypo-contact solvable Lie groups [De Andres, Fernandez, –, Ugarte].
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Proposition (Conti, –)

H: compact Lie group
ρ a representation of H on V.
Thene $H \ltimes_\rho V$ has a left-invariant contact structure if and only if $H \ltimes_\rho V$ is either $SU(2) \ltimes \mathbb{R}^4$ or $U(1) \ltimes \mathbb{C}$.

Then, if H is compact, the example $SU(2) \ltimes \mathbb{R}^4$ is unique in dimensions > 3.

If H is solvable we have

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There exists $H \ltimes \mathbb{R}^4$ admitting a contact $SU(3)$-structure whose associated spinor is generalized Killing if and only if the Lie algebra of H is isomorphic to one of the following

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Contact reduction

Let N^{2n+1} be a $(2n+1)$-dimensional manifold endowed with a contact metric structure (η, ϕ, g) and a spin structure compatible with the metric g and the orientation.

We say that a spinor ψ on N^{2n+1} is compatible if

$$\eta \cdot \psi = i^{2n+1}\psi, \quad \phi \cdot \psi = -ni\psi.$$

Suppose that S^1 acts on N^{2n+1} preserving both metric and contact form, so that the fundamental vector field X satisfies

$$\mathcal{L}_X \eta = 0 = \mathcal{L}_X \phi.$$

and denote by t its norm.

The moment map is given by $\mu = \eta(X)$.

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Assume that 0 is a regular value of μ and consider the hypersurface $\iota : \mu^{-1}(0) \to N^{2n+1}$.

Then the contact reduction is given by $N^{2n+1}//S^1 = \mu^{-1}(0)/S^1$ [Geiges, Willett].

• The contact $U(n)$-structure on N^{2n+1} induces a contact $U(n-1)$-structure on $N^{2n+1}//S^1$.

Let ν be the unit normal vector field, dual to the 1-form $i_{t^{-1}X}\phi$.

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Theorem (Conti, –)

N^{2n+1} with a contact $U(n)$-structure (g, η, ϕ) and a compatible generalized Killing spinor ψ.
Suppose that S^1 acts on N^{2n+1} preserving both structure and spinor and acts freely on $\mu^{-1}(0)$ with 0 regular value. Then the induced spinor ψ^π on $N^{2n+1}//S^1$ is generalized Killing if and only if at each point of $\mu^{-1}(0)$ we have

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where X is the fundamental vector field associated to the S^1-action, and t is the norm of X.

Example

If we apply the previous theorem to $SU(2) \ltimes \mathbb{R}^4$ we get a new hypo-contact structure on $S^2 \times \mathbb{T}^3$.

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